Influence functions and applications in inequality analysis

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Directional derivatives

- Let y be a random variable with distribution function $F(y)$ and continuous density function $F'(y) = f(y)$.
- Let $T(F)$ be a distributional statistic (e.g., a social evaluation function or indicator of a social outcome) that is qualitatively and infinitesimally robust. Examples: T(F) could be a τ-quantile, the mean or Gini coefficient.
- Let $T_n(x_1, ..., x_n) = T_n(F_n)$ be the sample estimator.
- If a functional T satisfies $\lim_{n\to\infty} T_n(F_n) = T(F)$ then it is called Fisher consistent at F.

Directional derivatives

- The inuence function (IF) is the directional derivative of $T(F)$ at F and it measures the effect of a small perturbation in F. It is also known as the Gâteaux derivative. Let H be some distribution other than F. When the data does not follow F exactly, but a slightly different distribution, one that is "going toward" H, the effect is revealed by the directional derivative of T at F in the direction of H:

$$\nabla T_{F \to H} = \frac{d}{dt} T(tH + (1-t)F)|_{t=0} = \lim_{t \to 0} \frac{T(tH + (1-t)F) - T(F)}{t}$$

- In other words, let x be an additional data point in a large sample that adds a perturbation the distribution with probability mass δ_x. H is then $H(y) = 1[x \geq y]$ and $h(y)$ is a density function with value 0 except at x. Then,

$$IF(x; T; F) = \lim_{t \to 0} \frac{T(t\delta_x + (1-t)F) - T(F)}{t}$$

- The IF of a distributional statistic (and hence of a social evaluation function) measures the relative effect of a small perturbation in the underlying outcome distribution on the statistic of interest. Within that approach and under the assumption that the distributional change in question is due to policy implementation, the IF of a social evaluation criterion may be viewed as a local measure of the distributional impact of policy.

Directional derivatives

- The IF satisfies $\int_{-\infty}^{\infty} IF(x; T; F)f(x)dx = 0$, that is, on average the influence of all perturbations should be 0.
- Then define the **recentered influence function** (RIF) as

$$RIF(x; T; F) = T(F) + IF(x; T; F)$$

We have that $E[RIF(x; T; F)] = E[T(F)] + \int_{-\infty}^{\infty} IF(x; T; F)f(x)dx = T(F)$.

Jacknife

- An empirical way to approximate IF is to use the so called *jacknife* method.
- Let T_n be the estimator using F_n, $\{y_1, y_2, \ldots, y_n\}$. Then the jacknife pseudo-value of observation i is
$T^*_{ni} = nT_n - (n-1)T_{n-1}(y_1, y_2, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) = nT_n - (n-1)T_{n-1}$.
Then to get IF we can use F_n instead of F and $-1/(n-1)$ for t, such that

$$IF(y_i; T; F_n) \cong \frac{T\left(\frac{n}{n-1}F_n - \frac{1}{n-1}\delta_{y_i}\right) - T(F_n)}{-1/(n-1)} = (n-1)(T_n - T_{n-1}) = T^*_{ni} - T_n$$

- This is a practical method to construct the IF functions when we cannot have an analytical solution.

IF and RIF of the mean

- Let y be a random variable with mean $\mu_F = E[y] = T_\mu(F)$.
- $$IF(T_\mu) = \nabla T_{\mu, F \to H} = \lim_{t \to 0} \frac{T_\mu(tH + (1-t)F) - T_\mu(F)}{t} = \mu_H - \mu_F$$
- $$RIF(x; T_\mu; F) = x - \mu_F$$

  The influence of an observation is as big as the observation itself.

IF and RIF of τ-quantile

- Let y be a random variable with τ-quantile, $\nu_\tau = T_{\nu_\tau}(F) = F^{-1}(\tau)$.
- $$IF(T_{\nu_\tau}) = [\tau - H(\nu_\tau)]/f(\nu_\tau)$$

  Proof: Let $w(t)$ be the τ-quntile of $tH + (1-t)F$, such that $tH(w(t)) + (1-t)F(w(t)) = \tau$ and also $w(0) = \nu_\tau$.

  $$\frac{d\tau}{dt} = 0 = H(w(t)) - F(w(t)) + w'(t)[th(w(t)) + (1-t)f(w(t))]$$

  such that

  $$w'(t) = \frac{F(w(t)) - H(w(t))}{th(w(t)) + (1-t)f(w(t))}.$$

  Therefore,

  $$IF(T_{\nu_\tau}) = \nabla T_{\nu_\tau, F\to H} = w'(0) = [\tau - H(\nu_\tau)]/f(\nu_\tau),$$

  $$RIF(x; T_{\nu_\tau}; F) = \nu_\tau + \frac{\tau - 1[y \le \nu_\tau]}{f(\nu_\tau)}.$$

IF and RIF of variance

- Let y be a random variable with variance $\sigma_2 = \int (y-\mu_F)^2 f(y)dy = \int (y-\mu_F)^2 dF$.
- $$IF(T_{\sigma_F^2}) = \sigma_H^2 - \sigma_F^2 + (\mu_H - \mu_F)^2$$

  Proof:

$$\frac{d}{dt}\int (y - t\mu_H - (1-t)\mu_F)^2[th(y) + (1-t)f(y)]dy\|_{t=0} = \int (y-\mu_F)^2(dH - dF) + (\mu_F - \mu_H)\int (y-\mu_F)^2 dF$$
$$= -\sigma_F^2 + \int (y-\mu_F)^2 dH = -\sigma_F^2 + \int (y - \mu_H + \mu_H + \mu_F)^2 dH = \sigma_H^2 - \sigma_F^2 + (\mu_H - \mu_F)^2$$

$$RIF(x; T_{\nu_\tau}; F) = (x - \mu_F)^2$$

IF and RIF of Gini

- Let y be a random variable with domain in $\mathbb{R}^+$. The Gini coefficient is a standard measure of inequality.

$$G_F = \frac{1}{2\mu_F}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)|x-y|dxdy = \frac{1}{\mu_F}\int F(y)(1-F(y)dy$$
$$= \frac{1}{2\mu_F}\int_0^1\int_0^1 |Q_y(\tau_1) - Q_y(\tau_1)|d\tau_1 d\tau_2$$

- $$IF(T_{G_F}) = \frac{\mu_F - \mu_H}{\mu_F}G_F + \frac{1}{\mu_F}\int [H(y) - F(y)][1-2F(y)]dy$$

Proof: Note that $\mu_F G_F = \int F(y)(1-F(y))dy$, then define $\mu_{tH+(1-t)F} G_{tH+(1-t)F} = \int tH(y) + (1-t)F(y)(1-tH(y)+(1-t)F(y))dy$. Take derivatives with respect to t, and evaluate at $t=0$:

$$\frac{d}{dt}[\mu_{tH+(1-t)F} G_{tH+(1-t)F}]|_{t=0} = (\mu_H - \mu_F)G_F + \mu_F IF(T_{G_F})$$

$$\frac{d}{dt}[\int tH(y) + (1-t)F(y)(1-tH(y)+(1-t)F(y))dy]|_{t=0} = \int [H(y)-F(y)][1-2F(y)]dy$$

IF and RIF of Gini

- $IF(x; T_G; F) = -\frac{\mu_F + x}{\mu_F} G_F + 1 - \frac{x}{\mu_F} + \frac{2}{\mu_F} \int_0^x F(y)dy$
- $RIF(x; T_G; F) = -\frac{x}{\mu_F} G_F + 1 - \frac{x}{\mu_F} + \frac{2}{\mu_F} \int_0^x F(y)dy$

IF and RIF of Poverty

- Let y be a random variable with domain in $\mathbb{R}^+$ and let z be the value that defines the poverty line (i is poor if $y_i \leq z$). The poverty rate is $P_F = \int_0^z F(y)dy$.
- $IF(T_P; F) = \frac{d}{dt} \int_0^z [th(y) + (1-t)f(y)]dy|_{t=0} = \int_0^z (dH - dF)$
- $RIF(x; T_P; F) = 1[x \leq z]$
- FGT(α) index (Foster, Greer, Thorbecke, 1984) of poverty is $T_{FGT_\alpha}(F) \int_0^z (1 - y/z)^\alpha f(y)dy$, con $\alpha \geq 0$.
- $IF(T_{FGT_\alpha}; F) = \frac{d}{dt} \int_0^z (1 - y/z)^\alpha [th(y) + (1-t)f(y)]dy|_{t=0} = \int_0^z (1 - y/z)^\alpha h(y)dy - T_{FGT_\alpha}$
- $RIF(x; T_{FGT_\alpha}; F) = (1 - x/z)^\alpha 1[x \leq z]$

RIF regression

- RIF regression method can be applied to conditional models, $Y|X$.
- We are interested in the effect of a marginal change in X. Let F_X be the distribution of X and G_X a perturbation. Define G_Y^* as the induced distribution of Y after that change in X.
- $$F_Y(y) = \int F_{Y|X}^{-1}(y|X=x) \cdot dF_X(x)$$ $$G_Y^*(y) = \int F_{Y|X}^{-1}(y|X=x) \cdot dG_X(x)$$
- $T(F_Y) = \int RIF(y;T;F_Y)dF_Y(y) = \int \int RIF(y;T;F_Y)dF_{Y|X)}(y|X = x)dF_X(x) = \int E[RIF(y;T;F_Y)|X=x]dF_X(x)$. Then,

$$\frac{dIF(F_Y \to (1-t)F_y + tG_Y^*)}{dt}|_{t=0} = \int E[RIF(y;T;F_Y)|X=x]d(F_X - G_X)(x).$$

- If X_j is continuous, then marginal values are

$$\int dE[RIF(y;T;F_Y)|X=x]/dx\ d(F_X)(x).$$

RIF regression

- Firpo, Fortin and Lemieux (2009) propose to run the models:

$$E[RIF(Y;T)] = X\beta + \epsilon$$

  for any distributional statistic $T(F_Y)$.
- The inuence function of a social evaluation criterion may be viewed as a local measure of the distributional impact of policy. This is analogous to what Rothe (2010) calls a **distributional policy effect**.

RIF regression

- Firpo, Fortin and Lemieux (2009) apply this analysis to the particular case of **unconditional quantile regression**.
- For quantiles, we cannot use the law of iterated expectations, i.e., $E[Y] = E\left[E(Y|X)\right]$.[1] That is,

$$F_Y^{-1}(\tau) \neq E_X[F_{Y|X}^{-1}(\tau|X),$$

  which means that we cannot recover the "unconditional quantile" from quantile regression.
- Thus, to study the effect of a change in X in they propose to run a regression of $\widehat{RIF(Y,\hat{q}_\tau)}$ on X, where $\hat{q}_\tau$ is the unconditional quantile of Y and $RIF()$ is the RIF of quantiles.

[1] Expectation is a linear operator. This result only applies to linear operators.

RIF regression

- Looking at the individual outcome as a function of policy and type (base on characteristics), one can resort to counterfactual decomposition of observed distributional change à la OaxacaBlinder to sort out the part that is due to policy and the part due to confounding factors.
- Let A and B be two groups (i.e., men vs. women, union vs. non-union workers, 2017 vs. 2018 individuals). Then,

$$T(F_Y^B) - T(F_Y^A) = \bar{X}_B(\beta_B - \beta_A) + (\bar{X}_B - \bar{X}_A)\beta_A,$$

where the first term corresponds to the explained **difference or composition or endowments effect** and the second to the **unexplained difference or price effect**.

RIF in STATA

- Download the package `rifreg.zip` in `https://faculty.arts.ubc.ca/nfortin/datahead.html`
- Copy files to the same directory you will use.
- `rifreg` Y X, `quantile(0.1)`
- `rifreg` Y X, `quantile(0.5)`
- `rifreg` Y X, `quantile(0.9)`
- `rifreg` Y X, `gini`
- `rifreg` Y X, `variance`

References

This slides are based on

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