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# Reduced form vector directional quantiles

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In this paper, we develop a reduced form multivariate quantile model, using a directional quantile framework. The proposed model is the solution to a collection of directional quantile models for a fixed orthonormal basis, in which each component represents a directional quantile that corresponds to a particular endogenous variable. The model thus delivers a map from the space of exogenous variables (or the $\sigma$-field generated by the information available at a particular time) and a unit ball whose dimension is given by the number of endogenous variables, to the space of endogenous variables. The main effect of interest is that of exogenous variables on the vector of endogenous variables, which depends on a multivariate quantile index. An estimator is proposed, using quantile regression time series models, and we study its asymptotic properties. The estimator is then applied to study the interdependence among countries in the European sovereign bonds credit default swap market.



## 1. Introduction

Lags occur in time series for several reasons, including price stickiness, psychological inertia, permanent vs. transitory shocks, adjustment costs, and delays in implementing new technologies. Modeling dynamic behavior has been a concern in econometrics, and constant-coefficient linear time-series models play a large role. Further, an important way to summarize the dynamics of macroeconomic data is to make use of a vector autoregressive (VAR) model. The VAR approach provides statistical tools for data description, forecasting, and structural inference to study rich dynamics in multivariate time-series models.

Nevertheless, the use of a constant-coefficient model as representative of time-series models may not be adequate, as these models ignore the effects that a succession of small and varied shocks may have on the structure of dynamic economic models, particularly for highly aggregated data series. Moreover, these models cannot appropriately account for the presence of asymmetric dynamic responses. Of particular interest is the asymmetric business cycle dynamics of economic variables, as the occurrence of asymmetries may call into question the usefulness of models with time invariant structures as means of modeling such series.

Quantile regression (QR) is a statistical method for estimating models of conditional quantile functions. This method offers a systematic strategy for examining how covariates influence the location, scale, and shape of the entire response distribution, thereby exposing a variety of heterogeneity in response dynamics. For a given cumulative distribution function $F_Y$ of a univariate random variable $Y$, the univariate quantile function is well defined. In particular, the $\tau$-quantile for

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$\tau \in (0, 1)$ is defined as $Q_Y(\tau) = \inf\{y \in \mathcal{Y} : \tau \leq F_Y(y)\}$, and if $F_Y$ is continuous, then $Q_Y(\tau) = F_Y^{-1}(\tau)$. In the multivariate case, however, there is no unique definition of a multivariate quantile function.

There is a growing literature on the estimation of QR models for multivariate random variables. Hallin et al. [14] and Paindaveine and Šiman [19,20] build on the definition of directional quantiles, whereby quantiles are equipped with a directional vector. Distributional features are thus explored by considering different directional models; see also [9] for related work. Wei [23] develops a bivariate quantile model, following the marginal-conditional structure of Ma and Koenker [18]. White et al. [24] develop an autoregressive model of the quantiles themselves, extending the CAViaR model of Engle and Manganelli [7] to the multivariate case. In related work, Han et al. [15] study quantile dependence among time-series models. Carlier et al. [3] propose a vector QR (linear) model that produces a monotone map, the gradient of a convex function. In a more general setup, Chernozhukov et al. [5] develop a concept of multivariate quantile based on transportation maps between a distribution of interest with a domain in multivariate real numbers and a unit ball of the same dimension. Finally, another approach is to use copula-based quantile models, as any multivariate distribution can be decomposed into its marginals and a dependence function or copula; see, e.g., [1,12,13]; however, such an approach requires imposing distributional assumptions.

The purpose of this paper is to generalize to the multivariate case the quantile autoregressive framework proposed by Koenker and Xiao [16] and Galvao et al. [11]. We develop a reduced form vector directional quantile (VDQ) model based on the multivariate directional quantiles of [14]. The definition of the VDQ model is based on a system of univariate directional quantiles, and, as such, it satisfies some of the monotonicity properties desired in a multivariate setting. We argue that this definition is natural in time-series contexts for which we are interested in estimating a reduced form model.

The proposed VDQ model is a solution to a collection of directional quantile models for a fixed orthonormal basis, in which each component represents a directional quantile that corresponds to a particular endogenous variable. The model thus delivers a map from the space of exogenous variables (or the $\sigma$-field generated by the information available at a particular time) and the unit ball whose dimension is given by the number of endogenous variables to the space of endogenous variables. The main effect of interest is that of exogenous variables on the endogenous variables vector, which depends on multivariate quantile indexes.

We apply the VDQ estimator to model European sovereign bonds interdependence. In particular, we propose a multivariate model for the sovereign bonds credit default swaps of Greece and Spain. We also study the effect of Euro-area monetary variables on those countries' sovereign bonds as a means to explore heterogeneity of monetary shocks.

The paper is organized as follows. Section 2 presents the theory of directional quantiles and the definition of the VDQ model. Section 3 provides the development of the case of a bivariate model. An investigation of its monotonicity properties is then given in Section 4. Section 5 describes the asymptotic theory. Section 6 presents an application of the VDQ to the European sovereign risk credit default swap market. Section 7 concludes.

## 2. Model

Consider an $m$-dimensional process $\boldsymbol{Y}_t = (Y_{1t}, \ldots, Y_{mt})^\top$ and assume that for all $t \in \{0, 1, \ldots\}$, $\boldsymbol{Y}_t \in \mathcal{Y} \subseteq \mathbb{R}^m$. Further, consider a $k \times 1$ vector of covariates $\boldsymbol{X}_t \in \mathcal{X} \subseteq \mathbb{R}^k$. Our goal is to develop a model for the conditional random variable $\boldsymbol{Y}_t|\boldsymbol{X}_t$. In particular, we seek to define and estimate the multivariate conditional quantile function of $\boldsymbol{Y}_t|\boldsymbol{X}_t$.

Of particular interest is the case of the covariates generated by the $\sigma$-field given by $(\boldsymbol{Y}_s : s < t)$ and all other information available at time $t$. One then deals with a vector autoregressive quantile model. For an autoregressive model of order $p$, $\boldsymbol{X}_t = (\boldsymbol{Y}_{t-1}^\top, \ldots, \boldsymbol{Y}_{t-p}^\top)^\top$ and $k = mp$, or, if we consider $d$ exogenous covariates $\boldsymbol{Z}_t$, then $\boldsymbol{X}_t = (\boldsymbol{Y}_{t-1}^\top, \ldots, \boldsymbol{Y}_{t-p}^\top, \boldsymbol{Z}_t^\top) \in \mathcal{Z} \subseteq \mathbb{R}^d$ and $k = mp + d$.

Let the vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_m)$ be an index of the $\mathbb{R}^m$ space, which is an element of the open unit ball in $\mathbb{R}^m$ (deprived of the origin) $\mathcal{T}^m = \{\boldsymbol{z} \in \mathbb{R}^m : 0 < \|\boldsymbol{z}\| < 1\}$, where $\|\cdot\|$ denotes the Euclidean norm. Our interest lies in defining and estimating

$$\boldsymbol{Q}_{\boldsymbol{Y}_t|\boldsymbol{X}_t}(\boldsymbol{\tau}|\boldsymbol{X}_t) = \boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{X}_t + \boldsymbol{A}(\boldsymbol{\tau}), \tag{1}$$

where $\boldsymbol{Q}$ is an $m \times 1$ vector, which corresponds to the multivariate quantiles of the $m$ random variables, $\boldsymbol{B}(\boldsymbol{\tau}) = (\boldsymbol{B}_1(\boldsymbol{\tau}), \ldots, \boldsymbol{B}_m(\boldsymbol{\tau}))^\top$ is an $m \times k$ matrix of coefficients with $\boldsymbol{B}_j(\boldsymbol{\tau})$ for each $j \in \{1, \ldots, m\}$, the corresponding $k \times 1$ vector of coefficients of the $j$th element in $\boldsymbol{Y}$, and $\boldsymbol{A}(\boldsymbol{\tau})$ is an $m \times 1$ vector of coefficients. Thus, $\boldsymbol{Q}$ is a map $\mathcal{X} \times \mathcal{T}^m \mapsto \mathcal{Y}$ and corresponds to our proposed definition of multivariate quantiles, the VDQs.

Our definition builds on the work of Hallin et al. [14], who propose to analyze the distributional features of multivariate response variables using the directional quantiles notion of [4,17,23] and others. Quantiles are analyzed in terms of a quantile magnitude and a direction. The vector $\boldsymbol{\tau}$ factorizes into $\boldsymbol{\tau} = \tau\boldsymbol{v}$, where $\tau = \|\boldsymbol{\tau}\| \in (0, 1)$ and $\boldsymbol{v} \in \{\boldsymbol{z} \in \mathbb{R}^m : \|\boldsymbol{z}\| = 1\}$. Then, $\tau$ represents a scalar quantile index, and $\boldsymbol{v}$ is an $m \times 1$ directional vector. We define $\boldsymbol{\Gamma}_{\boldsymbol{v}}$ as an $m \times (m-1)$-dimensional matrix, such that $(\boldsymbol{v}, \boldsymbol{\Gamma}_{\boldsymbol{v}})$ forms an orthonormal basis. Note that $\boldsymbol{\Gamma}_{\boldsymbol{v}}$ is not unique but, rather, any $m \times (m-1)$ matrix whose columns are orthogonal to $\boldsymbol{v}$ and to each other.

Following [14], we define the directional regression quantiles as the directional hyperplanes

$$\pi_{(\tau,\boldsymbol{v})} = \left\{(\boldsymbol{x}^\top, \boldsymbol{y}^\top)^\top \in \mathbb{R}^{k+m} : \boldsymbol{v}^\top\boldsymbol{y} = c(\tau, \boldsymbol{v}, \boldsymbol{\Gamma}_{\boldsymbol{v}})^\top\boldsymbol{\Gamma}_{\boldsymbol{v}}^\top\boldsymbol{y} + \boldsymbol{b}(\tau, \boldsymbol{v}, \boldsymbol{\Gamma}_{\boldsymbol{v}})^\top\boldsymbol{x} + a(\tau, \boldsymbol{v})\right\}$$

such that

$$\left\{\boldsymbol{c}(\tau,\boldsymbol{v},\boldsymbol{\Gamma_v})^\top,\boldsymbol{b}(\tau,\boldsymbol{v},\boldsymbol{\Gamma_v})^\top,a(\tau,\boldsymbol{v})\right\}^\top=\underset{(\boldsymbol{c}^\top,\boldsymbol{b}^\top,a)\in\mathbb{R}^{k+m}}{\arg\min}\ \mathrm{E}\left\{\rho_\tau(\boldsymbol{v}^\top\boldsymbol{Y}_t-\boldsymbol{c}^\top\boldsymbol{\Gamma_v}^\top\boldsymbol{Y}_t-\boldsymbol{b}^\top\boldsymbol{X}_t-a)\right\},$$

where $\rho_\tau(z)=|z|+(2\tau-1)z$ is the QR (univariate) check function. In this case, $a$ is a scalar, $\boldsymbol{b}$ is a $k\times 1$ vector, and $\boldsymbol{c}$ is an $(m-1)\times 1$ vector. Note that, while $a$ depends on $\tau$ and $\boldsymbol{v}$, $\boldsymbol{c}$ and $\boldsymbol{b}$ depend on the triplet $(\tau,\boldsymbol{v},\boldsymbol{\Gamma_v})$; see [14], p. 639. Note, however, that, for fixed $(\boldsymbol{v},\boldsymbol{\Gamma_v})$, the directional quantiles are univariate QR models. That is, for fixed $\boldsymbol{v}$, $\boldsymbol{v}^\top\boldsymbol{Y}_t$ is a scalar. Then, for fixed $(\boldsymbol{v},\boldsymbol{\Gamma_v})$, this representation implicitly defines a conditional quantile model of the response variable $\boldsymbol{v}^\top\boldsymbol{Y}_t$, conditional on $\boldsymbol{W}_t^{\boldsymbol{\Gamma_v}}=\left\{(\boldsymbol{\Gamma_v}^\top\boldsymbol{Y}_t)^\top,\boldsymbol{X}_t^\top\right\}^\top$, i.e.,

$$Q_{\boldsymbol{v}^\top\boldsymbol{Y}_t|\boldsymbol{W}_t^{\boldsymbol{\Gamma_v}}}(\tau|\boldsymbol{W}_t^{\boldsymbol{\Gamma_v}})=\boldsymbol{c}_{(\boldsymbol{v},\boldsymbol{\Gamma_v})}(\tau)^\top\boldsymbol{\Gamma_v}^\top\boldsymbol{Y}_t+\boldsymbol{b}_{(\boldsymbol{v},\boldsymbol{\Gamma_v})}(\tau)^\top\boldsymbol{X}_t+a_{\boldsymbol{v}}(\tau).\qquad(2)$$

Our objective is to estimate Eq. (1), using a system of equations given by directional quantiles as described above. We fix an orthonormal basis $\mathcal{V}^m=\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_m\}$ and define the corresponding orthogonal partitions $(\boldsymbol{v}_j,\boldsymbol{\Gamma}_{\boldsymbol{v}_j})$ indexed by $j\in\{1,\ldots,m\}$. For further reference, we use the $-j$ notation to denote a vector or matrix that excludes the $j$th column or row. Moreover, we define $\boldsymbol{v}_{-j}=\boldsymbol{\Gamma}_{\boldsymbol{v}_j}$. In particular, we let $\{\boldsymbol{v}_j\}_{j=1}^m$ be column vectors with 1 in the $j$th component and 0 otherwise, while $\{\boldsymbol{v}_{-j}\}_{j=1}^m$ are $m\times(m-1)$ matrices that exclude the $j$th column from $\mathcal{V}^m$. Then, we have, for each $j$, that $\boldsymbol{v}_j^\top\boldsymbol{Y}_t=Y_{jt}$ and $\boldsymbol{v}_{-j}^\top\boldsymbol{Y}_t=\boldsymbol{Y}_{-jt}=(Y_{1t},\ldots,Y_{j-1t},Y_{j+1t},\ldots,Y_{mt})^\top$.

The directional quantiles are key components in our definition. For the fixed orthonormal basis $\mathcal{V}^m$, the objects in Eq. (2) are of the form $Q_{Y_{jt}|\boldsymbol{W}_t^j}(\tau_j|\boldsymbol{W}_t^j)$ with $j\in\{1,\ldots,m\}$ and $\boldsymbol{W}_t^j=\left(\boldsymbol{Y}_{-jt}^\top,\boldsymbol{X}_t^\top\right)^\top$, and correspond to the $\tau_j$-quantile of $Y_{jt}$ conditional on $\boldsymbol{Y}_{-jt}$, the contemporaneous variables of the other $m-1$ elements in $\boldsymbol{Y}_t$, and the lags and exogenous variables in $\boldsymbol{X}_t$. We then solve for all $j$ simultaneously at particular quantile indexes given by the vector $\boldsymbol{\tau}$. That is, the definition of the VDQ model requires us to compute the $\tau_j$-quantile of $Y_{jt}$ conditional on the other $-j$ components evaluated in the given $\boldsymbol{\tau}_{-j}$ quantiles, which, in turn, depend on the value of the $\tau_j$-quantile of the $j$th component. Thus, we need to simultaneously solve a system of equations, such that the $\boldsymbol{\tau}$ quantiles are obtained.

We then define the VDQ as $\boldsymbol{Q}(\boldsymbol{\tau},\boldsymbol{x}_t)=(Q_1(\boldsymbol{\tau},\boldsymbol{x}_t),\ldots,Q_m(\boldsymbol{\tau},\boldsymbol{x}_t))^\top$, for $\boldsymbol{X}_t=\boldsymbol{x}_t$, where

$$\begin{cases}Q_1(\boldsymbol{\tau},\boldsymbol{x}_t)&=Q_{Y_{1t}|\boldsymbol{Y}_{-1t},\boldsymbol{X}_t}\{\tau_1|\boldsymbol{Y}_{-1t}=\boldsymbol{Q}_{-1}(\boldsymbol{\tau},\boldsymbol{x}_t),\boldsymbol{X}_t=\boldsymbol{x}_t\}\\ \quad\vdots&=\vdots\\ Q_m(\boldsymbol{\tau},\boldsymbol{x}_t)&=Q_{Y_{mt}|\boldsymbol{Y}_{-mt},\boldsymbol{X}_t}\{\tau_m|\boldsymbol{Y}_{-mt}=\boldsymbol{Q}_{-m}(\boldsymbol{\tau},\boldsymbol{x}_t),\boldsymbol{X}_t=\boldsymbol{x}_t\},\end{cases}\qquad(3)$$

with $\boldsymbol{Q}_{-j}(\boldsymbol{\tau},\boldsymbol{x}_t)=(Q_1(\boldsymbol{\tau},\boldsymbol{x}_t),\ldots,Q_{j-1}(\boldsymbol{\tau},\boldsymbol{x}_t),Q_{j+1}(\boldsymbol{\tau},\boldsymbol{x}_t),\ldots,Q_m(\boldsymbol{\tau},\boldsymbol{x}_t))^\top$, i.e., leaving the $j$th component out.

Then, assuming linear conditional directional quantile functions, as in Eq. (2), we define the reduced form VDQ model of the $m$-dimensional process $\boldsymbol{Y}_t|\boldsymbol{X}_t$ in the following way:

$$\begin{cases}Q_1(\boldsymbol{\tau},\boldsymbol{x}_t)&=\boldsymbol{c}_1(\tau_1)^\top\boldsymbol{Q}_{-1}(\boldsymbol{\tau},\boldsymbol{x}_t)+\boldsymbol{b}_1(\tau_1)^\top\boldsymbol{x}_t+a_1(\tau_1)\\ \quad\vdots&=\vdots\\ Q_m(\boldsymbol{\tau},\boldsymbol{x}_t)&=\boldsymbol{c}_m(\tau_m)^\top\boldsymbol{Q}_{-m}(\boldsymbol{\tau},\boldsymbol{x}_t)+\boldsymbol{b}_m(\tau_m)^\top\boldsymbol{x}_t+a_m(\tau_m),\end{cases}\qquad(4)$$

where $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ and $\{\boldsymbol{b}_j(\tau_j)\}_{j=1}^m$ are vectors of dimension $(m-1)\times 1$ and $k\times 1$, respectively, and $\{a_j(\tau_j)\}_{j=1}^m$ are scalars.

The following definition provides a summary of the object of interest.

**Definition 1** (*Vector Directional Quantile (VDQ)*)**.** We define the following matrices based on the coefficients of Eq. (4): $\boldsymbol{C}(\boldsymbol{\tau})=(\boldsymbol{C}_1(\tau_1),\ldots,\boldsymbol{C}_m(\tau_m))^\top$ is an $m\times m$ matrix in which the $m\times 1$ vectors $\{\boldsymbol{C}_j(\tau_j)\}_{j=1}^m$ contain all the elements of the $m-1$ vector of coefficients $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ augmented with a 0 in the corresponding $j$th component, $\boldsymbol{b}(\boldsymbol{\tau})=(\boldsymbol{b}_1(\tau_1),\ldots,\boldsymbol{b}_m(\tau_m))^\top$ is an $m\times k$ matrix, and $\boldsymbol{a}(\boldsymbol{\tau})=(a_1(\tau_1),\ldots,a_m(\tau_m))^\top$ is an $m\times 1$ vector. Then

$$\boldsymbol{Q}(\boldsymbol{\tau},\boldsymbol{x}_t)=\{\boldsymbol{I}_m-\boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\{\boldsymbol{b}(\boldsymbol{\tau})\boldsymbol{x}_t+\boldsymbol{a}(\boldsymbol{\tau})\}=\boldsymbol{B}(\boldsymbol{\tau})\boldsymbol{x}_t+\boldsymbol{A}(\boldsymbol{\tau}),\qquad(5)$$

where $\boldsymbol{I}_m$ is the $m$-dimensional identity matrix, $\boldsymbol{B}(\boldsymbol{\tau})=\{\boldsymbol{I}_m-\boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{b}(\boldsymbol{\tau})$ and $\boldsymbol{A}(\boldsymbol{\tau})=\{\boldsymbol{I}_m-\boldsymbol{C}(\boldsymbol{\tau})\}^{-1}\boldsymbol{a}(\boldsymbol{\tau})$.

The following assumption guarantees that there is a solution to the system of equations defined above.

**Assumption 1.** For all $\boldsymbol{\tau}\in(0,1)^m$, $\boldsymbol{I}_m-\boldsymbol{C}(\boldsymbol{\tau})$ is non-singular.

Note that the proposed VDQ model is a fixed point of a system of equations given by (3). In this case, the model requires the running of $m$ separate QR models of each of the $m$ elements in $\boldsymbol{Y}$ with respect to the other $m-1$ components and $\boldsymbol{X}$, and then solving for a system of equations of $m$ unknowns, given by $\boldsymbol{Q}(\boldsymbol{\tau},\boldsymbol{x}_t)$, and $m$ equations, each from a different directional quantile model. If linearity of the directional quantiles is assumed, the solution simplifies to that in Definition 1, in which case the $m$ equations are given by (4).

## 3. Intuition for our proposed estimator: Bivariate model

To motivate our proposed methodology, consider the following bivariate example. Let $\boldsymbol{Y}_t = (Y_{1t}, Y_{2t})^\top$ be the endogenous variables and let $X_t$ be an exogenous scalar random variable. Consider the following bivariate process

$$Y_{1t} = \gamma_1 Y_{2t} + \beta_1 X_t + \alpha_1 + \epsilon_{1t},\ \epsilon_{1t} \sim \text{iid}(0, \sigma^2_{1t}),$$
$$Y_{2t} = \gamma_2 Y_{1t} + \beta_2 X_t + \alpha_2 + \epsilon_{2t},\ \epsilon_{2t} \sim \text{iid}(0, \sigma^2_{2t}). \quad (6)$$

The parameters $(\gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2)$ are the so-called structural parameters. These cannot be identified unless additional assumptions are made, as in structural VAR models. The following calculations show that, by simultaneously considering the conditional models $Y_1|(Y_2, X)$ and $Y_2|(Y_1, X)$, we can solve for the reduced form in a standard mean VAR, i.e., $\text{E}(\boldsymbol{Y}_t|X_t)$. This is, in turn, the logic behind our VDQ definition, indexed by a pair $\boldsymbol{\tau} = (\tau_1, \tau_2) \in (0, 1)^2$, which describes the bivariate distribution of $\boldsymbol{Y}_t$ conditional on $X_t$, which is, in fact, a reduced form model. The conditional expectations $\text{E}(Y_{1t}|X_t = x)$, $\text{E}(Y_{2t}|X_t = x)$, i.e., the reduced form mean VAR model, can be identified and consistently estimated by using the model

$$\text{E}(Y_{1t}|X_t = x) = \frac{\beta_1 + \gamma_1\beta_2}{1-\gamma_1\gamma_2}x + \frac{\alpha_1 + \gamma_1\alpha_2}{1-\gamma_1\gamma_2}, \quad \text{E}(Y_{2t}|X_t = x) = \frac{\beta_2 + \gamma_2\beta_1}{1-\gamma_1\gamma_2}x + \frac{\alpha_2 + \gamma_2\alpha_1}{1-\gamma_1\gamma_2}.$$

Note that this can be found by solving a system of equations, using

$$\text{E}\,(Y_{1t}|X_t = x) = \gamma_1 \text{E}(Y_{2t}|X_t = x) + \beta_1 x + \alpha_1 + \text{E}(\epsilon_{1t}|X_t = x),$$
$$\text{E}\,(Y_{2t}|X_t = x) = \gamma_2 \text{E}(Y_{1t}|X_t = x) + \beta_1 x + \alpha_2 + \text{E}(\epsilon_{2t}|X_t = x),$$

where $\text{E}(\epsilon_j|X_t = x) = \text{E}(\epsilon_j) = 0$ for $j \in \{1, 2\}$. In other words, the reduced form can be obtained by evaluating the corresponding conditional expectations (where we are conditioning on $X$ only) and solving.

Consider now the conditional form model

$$\text{E}(Y_{1t}|Y_{2t}, X_t) = c_1 Y_{2t} + b_1 X_t + a_1, \quad \text{E}(Y_{2t}|Y_{1t}, X_t) = c_2 Y_{1t} + b_2 X_t + a_2,$$

and note that, by adding the endogenous variables, i.e., $Y_{-j}$, to the model above, it produces another model for which, in general, $a_j \neq \alpha_j$, $b_j \neq \beta_j$, $c_j \neq \gamma_j$, $j \in \{1, 2\}$. The conditional model could thus be interpreted as a biased structural system, provided that $(\gamma_1, \gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2)$ may not be recovered. This model could be interpreted in terms of the directional models defined in the previous section, where we set $\boldsymbol{v}_1 = (1, 0)^\top$ and $\boldsymbol{v}_2 = (0, 1)^\top$, such that $\boldsymbol{\Gamma}_{\boldsymbol{v}_1} = \boldsymbol{v}_{-1} = (0, 1)^\top$ and $\boldsymbol{\Gamma}_{\boldsymbol{v}_2} = \boldsymbol{v}_{-2} = (1, 0)^\top$. Then, the orthonormal basis is $\mathcal{V}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We evaluate $\boldsymbol{Y}$ on a given direction $\boldsymbol{v}_j$, $j \in \{1, 2\}$, by considering the model $\boldsymbol{v}_j^\top \boldsymbol{Y} = Y_j$, conditional on $\boldsymbol{v}_{-j}^\top \boldsymbol{Y} = Y_{3-j}$ and $X$.

A simple result is that the reduced form model can be obtained, as an unnecessary detour, using the conditional form. That is, for $j \in \{1, 2\}$,

$$\frac{a_j + c_j a_{3-j}}{1 - c_j c_{3-j}} = \frac{\alpha_j + \gamma_j \alpha_{3-j}}{1 - \gamma_j \gamma_{3-j}}, \quad \frac{b_j + c_j b_{3-j}}{1 - c_j c_{3-j}} = \frac{\beta_j + \gamma_j \beta_{3-j}}{1 - \gamma_j \gamma_{3-j}}.$$

We consider whether this result could be generalized to quantiles. As a note of caution, we explain that, contrary to the expectation operator, quantiles are not linear operators. We do, however, consider a definition of multivariate quantiles that is built upon conditional forms. Consider the estimation of the conditional QR models

$$q_1(\tau_1, y_2, x) = Q_{Y_1|(Y_2,X)}(\tau_1|y_2, x) = c_1(\tau_1)y_2 + b_1(\tau_1)x + a_1(\tau_1),$$
$$q_2(\tau_2, y_1, x) = Q_{Y_2|(Y_1,X)}(\tau_2|y_1, x) = c_2(\tau_2)y_1 + b_2(\tau_2)x + a_2(\tau_2),$$

where

$$Q_{Y_j|(Y_{3-j},X)}(\tau_j|y_{3-j}, x) = \inf\{y_j \in \mathcal{Y}_j : \tau_j \le F_{Y_j|(Y_{3-j},X)}(y_j|y_{3-j}, x)\},$$

where $F_{Y_j|(Y_{3-j},X)}(y_j|y_{3-j}, x)$ is the conditional distribution function of $Y_j$ conditional on $(Y_{3-j}, X)$, for $j \in \{1, 2\}$, and $\boldsymbol{\tau} = (\tau_1, \tau_2) \in (0, 1)^2$. This corresponds to the directional quantiles for the fixed orthonormal basis $\mathcal{V}^2$ defined above.

Each equation is seen as a particular directional quantile, as in [14]. As such, they provide useful information about the joint distribution of $(Y_1, Y_2)^\top$. However, the parameters $\{c_j(\tau_j), b_j(\tau_j), a_j(\tau_j)\}$, $j \in \{1, 2\}$, do not have a structural interpretation, and additional exclusion restrictions are needed to identify the corresponding structural parameters; see [6,18]. Nevertheless, they are valid models when the conditional quantiles are linear, and they serve our purpose of estimating a reduced form multivariate model.

We now set the system of equations to solve for $\boldsymbol{Q}(\boldsymbol{\tau}, x) = (Q_1(\boldsymbol{\tau}, x), Q_2(\boldsymbol{\tau}, x))^\top$, defined as

$$Q_1(\boldsymbol{\tau}, x) = Q_1\,[\tau_1, y_2 = q_2\{\tau_2, Q_1(\boldsymbol{\tau}, x), x\}, x] = c_1(\tau_1)Q_2(\boldsymbol{\tau}, x) + b_1(\tau_1)x + a_1(\tau_1),$$
$$Q_2(\boldsymbol{\tau}, x) = Q_2\,[\tau_2, y_1 = q_1\{\tau_1, Q_2(\boldsymbol{\tau}, x), x\}, x] = c_2(\tau_2)Q_1(\boldsymbol{\tau}, x) + b_2(\tau_2)x + a_2(\tau_2).$$

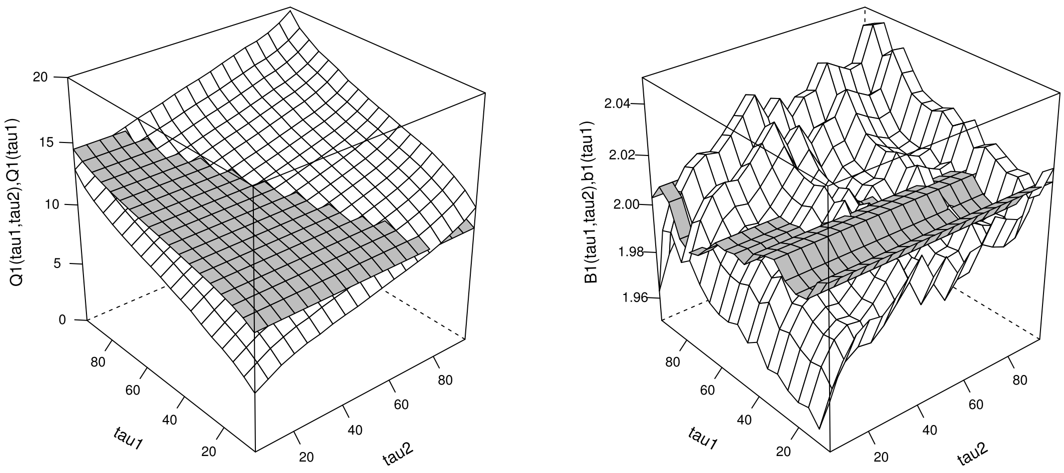


**Fig. 1.** Bivariate Example 1. Notes: Data generating process: $(Y_{1t}, Y_{2t}) = (\gamma_1 Y_{2t} + \beta_1 X_t + \alpha_1 + \epsilon_{1t}, \gamma_2 Y_{1t} + \beta_2 X_t + \alpha_2 + \epsilon_{2t})$, $\epsilon_{jt} \sim$ iid $\mathcal{N}(0, 1)$, $X_t \sim$ iid $\mathcal{U}(0, 10)$ for $t \in \{1, \ldots, T\}$, $\alpha_j = \beta_j = 1$ and $\gamma_j = 0.5$, for $j \in \{1, 2\}$, $T = 10{,}000$. Left panel: The white surface is $\hat{Q}_1^{VDQ}\{(\tau_1, \tau_2), x = 5\} = \hat{A}_1(\tau_1, \tau_2) + \hat{B}_1(\tau_1, \tau_2) \times 5$, the estimated bivariate VDQ for $Y_1$ evaluated at $x = 5$, and the gray surface is $\hat{Q}_1^{QR}(\tau_1, x = 5) = \hat{\alpha}(\tau_1) + \hat{\beta}_1(\tau_1) \times 5$, the predicted $\tau_1$-quantile of a QR of $Y_1$ on $X$ evaluated at $x = 5$. Right panel: The white surface is $\hat{B}_1(\tau_1, \tau_2)$, and the gray surface is $\hat{\beta}_1(\tau_1)$.

The solution is thus

$$Q_1(\boldsymbol{\tau}, x) = \frac{b_1(\tau_1) + c_1(\tau_1) b_2(\tau_2)}{1 - c_1(\tau_1) c_2(\tau_2)} x + \frac{a_1(\tau_1) + c_1(\tau_1) a_2(\tau_2)}{1 - c_1(\tau_1) c_2(\tau_2)} = B_1(\boldsymbol{\tau}) x + A_1(\boldsymbol{\tau}),$$
$$Q_2(\boldsymbol{\tau}, x) = \frac{b_2(\tau_2) + c_2(\tau_2) b_1(\tau_1)}{1 - c_1(\tau_1) c_2(\tau_2)} x + \frac{a_2(\tau_2) + c_2(\tau_2) a_1(\tau_1)}{1 - c_1(\tau_1) c_2(\tau_2)} = B_2(\boldsymbol{\tau}) x + A_2(\boldsymbol{\tau}),$$

where, for $j \in \{1, 2\}$,

$$B_j(\boldsymbol{\tau}) = \frac{b_j(\tau_j) + c_j(\tau_j) b_{3-j}(\tau_{3-j})}{1 - c_1(\tau_1) c_2(\tau_2)}, \quad A_j(\boldsymbol{\tau}) = \frac{a_j(\tau_j) + c_j(\tau_j) a_{3-j}(\tau_{3-j})}{1 - c_1(\tau_1) c_2(\tau_2)}.$$

Then, by assuming linearity of the conditional (directional) quantile models, the above solution is valid. Our definition of VDQ for $m = 2$ is thus given by $Q_1(\boldsymbol{\tau}, x)$ and $Q_2(\boldsymbol{\tau}, x)$. Note that we are, in fact, estimating a reduced form model, as we are modeling $\boldsymbol{Y}_t | X_t$, as in Eq. (1).

To illustrate the VDQ model, we implement two simulations of the bivariate example given by the data-generating process (6) above. Let $\epsilon_{jt} \sim$ iid $\mathcal{N}(0, \sigma_{jt}^2)$, $X_t \sim$ iid $\mathcal{U}(0, 10)$ for all $t \in \{1, \ldots, T\}$, and set $\alpha_j = \beta_j = 1$ and $\gamma_j = 0.5$, for $j = 1, 2$. Example 1 imposes homoscedastic innovations, with $\sigma_{jt}^2 = 1$ for all $t$ and $j \in \{1, 2\}$. Example 2 has heteroscedastic innovations, with $\sigma_{jt}^2 = (1+0.5X_t)^2$ for all $t$ and $j \in \{1, 2\}$. We consider random samples of size $T = 10{,}000$. We then compute the VDQ estimator for $(\tau_1, \tau_2) \in (0.05, \ldots, 0.95)^2$, described in Section 5, and compare it with univariate standard QR.

Figs. 1 (left panel) and 2 (left panel) plot, for Examples 1 and 2, respectively,

$$\hat{Q}_1^{VDQ}\{(\tau_1, \tau_2), x = 5\} = \hat{B}_1(\tau_1, \tau_2) \times 5 + \hat{A}_1(\tau_1, \tau_2),$$

the estimated bivariate VDQ for $Y_1$ evaluated at $x = 5$, and

$$\hat{Q}_1^{QR}(\tau_1, x = 5) = \hat{\beta}_1(\tau_1) \times 5 + \hat{\alpha}(\tau_1),$$

the predicted $\tau_1$-quantile of a QR of $Y_1$ on $X$ evaluated at $x = 5$, both as a function of $(\tau_1, \tau_2)$. Figs. 1 (right panel) and 2 (right panel) plot, for Examples 1 and 2, respectively, $\hat{B}_1(\tau_1, \tau_2)$, the estimated VDQ coefficients for the effect of $X$ on $Y_1$, and $\hat{\beta}_1(\tau_1)$, the estimated QR coefficient for the effect of $X$ on $Y_1$, also as a function of $(\tau_1, \tau_2)$. Note that, by construction, $\hat{Q}_1^{QR}(\tau_1, x = 5)$ and $\hat{\beta}_1(\tau_1)$ do not vary with $\tau_2$.

In both cases, the VDQ model has a larger variation than does the univariate model across $\tau_1$ and $\tau_2$. This determines that greater flexibility can be obtained by considering a bivariate VDQ model rather than the standard univariate model. Note, however, that there are differences between the homoscedastic and heteroscedastic examples in terms of the estimated effects of $X$ on $Y_1$. In the former model, the estimated slope coefficients, $\hat{B}_1(\tau_1, \tau_2)$ and $\hat{\beta}_1(\tau_1)$ are close to 2 for all $\tau_1$ and $\tau_2$. In the latter model, $\hat{B}_1(\tau_1, \tau_2)$ varies across $\tau_1$ and $\tau_2$, while $\hat{\beta}_1(\tau_1)$ varies across $\tau_1$, but $\hat{B}_1(\tau_1, \tau_2)$ offers more variation than $\hat{\beta}_1(\tau_1)$.

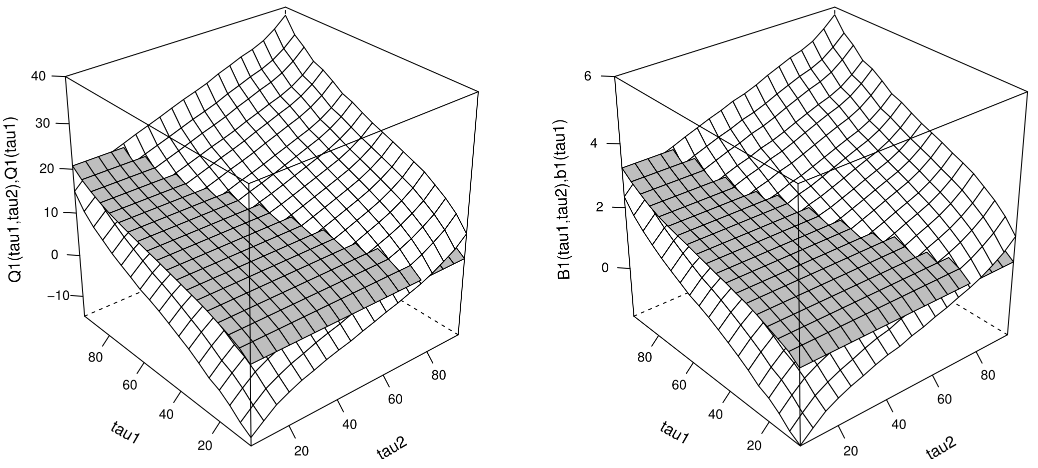


**Fig. 2.** Bivariate Example 2. Notes: Data generating process: $(Y_{1t}, Y_{2t}) = (\gamma_1 Y_{2t} + \beta_1 X_t + \alpha_1 + \epsilon_{1t}, \gamma_2 Y_{1t} + \beta_2 X_t + \alpha_2 + \epsilon_{2t})$, $\epsilon_{jt} \sim$ iid $\mathcal{N}\{0, (1 + 0.5X_t)\}$, $X_t \sim$ iid $\mathcal{U}(0, 10)$ for $t \in \{1, \ldots, T\}$, $\alpha_j = \beta_j = 1$ and $\gamma_j = 0.5$, for $j \in \{1, 2\}$, $T = 10{,}000$. Left panel: The white surface is $\hat{Q}_1^{VDQ}\{(\tau_1, \tau_2), x = 5\} = \hat{A}_1(\tau_1, \tau_2) + \hat{B}(\tau_1, \tau_2) \times 5$, the estimated bivariate VDQ for $Y_1$ evaluated at $x = 5$, and the gray surface is $\hat{Q}_1^{QR}(\tau_1, x = 5) = \hat{\alpha}(\tau_1) + \hat{\beta}_1(\tau_1) \times 5$, the predicted $\tau_1$-quantile of a QR of $Y_1$ on $X$ evaluated at $x = 5$. Right panel: The white surface is $\hat{B}(\tau_1, \tau_2)$, and the gray surface is $\hat{\beta}_1(\tau_1)$.

## 4. Monotonicity properties

This section concerns the requirements to satisfy non-decreasing monotonicity in the VDQ model. Let $\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) = \boldsymbol{b}(\boldsymbol{\tau})\boldsymbol{x} + \boldsymbol{a}(\boldsymbol{\tau})$ and $\boldsymbol{\Lambda}(\boldsymbol{\tau}) = \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1}$. Following [3], monotonicity in multivariate quantile models is defined with respect to $\boldsymbol{\tau}$ by the condition

$$\forall_{\boldsymbol{\tau}, \boldsymbol{\tau}' \in (0,1)^m}\ \forall_{\boldsymbol{x} \in \mathbb{R}^k}\quad \left\{\boldsymbol{\Lambda}(\boldsymbol{\tau})\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Lambda}(\boldsymbol{\tau}')\boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') \geq 0. \tag{7}$$

This can be written in two terms as

$$\forall_{\boldsymbol{\tau}, \boldsymbol{\tau}' \in (0,1)^m}\ \forall_{\boldsymbol{x} \in \mathbb{R}^k}\quad \left\{\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top \boldsymbol{\Lambda}(\boldsymbol{\tau})^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') + \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})^\top \left\{\boldsymbol{\Lambda}(\boldsymbol{\tau}) - \boldsymbol{\Lambda}(\boldsymbol{\tau}')\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') \geq 0.$$

Note that, because $\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x})$ is obtained separately from each equation, it depends only on the corresponding $\tau_j$-quantile for each $j$ row of $\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x})$. This is, however, only one component in the quantile system, and it excludes the contemporaneous interdependence of the $Y$s, i.e., it does not take into consideration the $\{\boldsymbol{c}_j(\tau_j)\}_{j=1}^m$ coefficients. If $\boldsymbol{\Lambda}(\boldsymbol{\tau}) = \boldsymbol{I}_m$, i.e., there is no contemporaneous interdependence, and $\boldsymbol{c}_j(\tau_j) = \boldsymbol{0}_{m-1}$ for all $j$, then the model satisfies monotonicity by definition of the univariate quantile models, as

$$\begin{aligned}\left\{\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') &= \sum_{j=1}^m \{\boldsymbol{b}_j(\tau_j)^\top \boldsymbol{x} + a_j(\tau_j) - \boldsymbol{b}_j(\tau_j')^\top \boldsymbol{x} - a_j(\tau_j')\}(\tau_j - \tau_j') \\ &= \sum_{j=1}^m \{Q_{Y_j|(\boldsymbol{Y}_{-j}, \boldsymbol{X})}(\tau_j | \boldsymbol{Y}_{-j}, \boldsymbol{X} = \boldsymbol{x}) - Q_{Y_j|(\boldsymbol{Y}_{-j}, \boldsymbol{X})}(\tau_j' | \boldsymbol{Y}_{-j}, \boldsymbol{X} = \boldsymbol{x})\}(\tau_j - \tau_j') \geq 0.\end{aligned}$$

In general, we impose the following assumption:

**Assumption 2.** For all $\boldsymbol{\tau}, \boldsymbol{\tau}' \in (0, 1)^m$, $\boldsymbol{x} \in \mathbb{R}^k$, $\left\{\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') \geq 0$.

Assumption 2 determines that for each $j \in \{1, \ldots, m\}$, the directional quantile model $Q_{Y_{jt}|\boldsymbol{Y}_{-jt}, \boldsymbol{X}_t}(\tau | \boldsymbol{Y}_{-jt}, \boldsymbol{X}_t)$ satisfies monotonicity in the last components, i.e., $\boldsymbol{b}_j(\tau)^\top \boldsymbol{X}_t + a_j(\tau)$, for all possible values of the first component, i.e., $\boldsymbol{c}_j(\tau)^\top \boldsymbol{Y}_{-jt}$.

If $\boldsymbol{\Lambda}(\boldsymbol{\tau}) = \boldsymbol{\Lambda}$ for all $\boldsymbol{\tau} \in (0, 1)^m$, i.e., a constant matrix that does not depend on $\boldsymbol{\tau}$, monotonicity depends on the first term $\left\{\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top \boldsymbol{\Lambda}(\boldsymbol{\tau})^\top (\boldsymbol{\tau} - \boldsymbol{\tau}')$. Assumption 1 determines that $\boldsymbol{C}(\tau)$ is a convergent matrix. A mild refinement, such as $\boldsymbol{I}_m - \boldsymbol{C}(\tau)$ as an M-matrix, i.e., $\boldsymbol{C}(\tau) \geq \boldsymbol{0}$, determines that $\boldsymbol{\Lambda}(\tau)$ is a positive matrix. As such, if each element in $\boldsymbol{\tau} - \boldsymbol{\tau}'$ has the same sign, then monotonicity is satisfied.

For a more general case, we consider the following assumption:

**Assumption 3.**

(i) For all $\boldsymbol{\tau} \in (0, 1)^m$, let $\boldsymbol{\Lambda}(\boldsymbol{\tau}) = \{\boldsymbol{I}_m - \boldsymbol{C}(\boldsymbol{\tau})\}^{-1} \in \mathcal{L}$, where $\mathcal{L}$ is the class of two-way semi-monotone matrices, such that for all $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathbb{R}^m$, $\boldsymbol{\lambda}_1^\top \boldsymbol{\lambda}_2 \geq 0$ implies $\boldsymbol{\lambda}_1^\top \boldsymbol{\Lambda}^\top \boldsymbol{\lambda}_2 \geq 0$ for all $\boldsymbol{\Lambda} \in \mathcal{L}$.

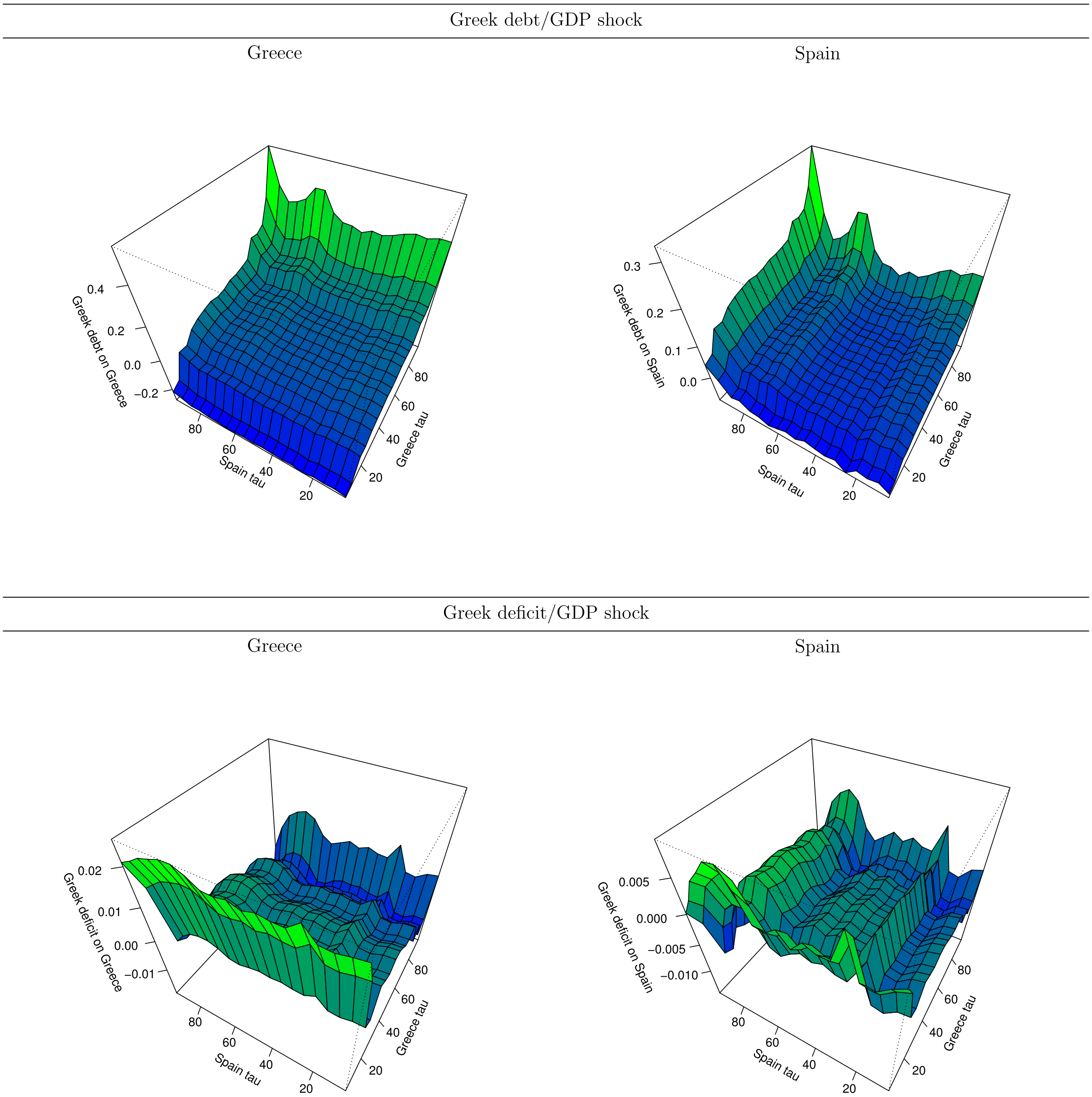


**Fig. 3.** Greek debt/GDP and deficit/GDP effects on 10-year CDS, VDQ coefficients.

(ii) For all $\boldsymbol{\tau}, \boldsymbol{\tau}' \in (0, 1)^m$, $\boldsymbol{x} \in \mathbb{R}^k$, $\boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})^\top \left\{\boldsymbol{\Lambda}(\boldsymbol{\tau}) - \boldsymbol{\Lambda}(\boldsymbol{\tau}')\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}') \geq 0$.

Assumption 3(i) implies that $\boldsymbol{\Lambda}(\boldsymbol{\tau})$ does not change the sign of $\left\{\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x}) - \boldsymbol{\Pi}(\boldsymbol{\tau}', \boldsymbol{x})\right\}^\top (\boldsymbol{\tau} - \boldsymbol{\tau}')$, which is assumed to be non-negative by Assumption 2. Intuitively, this condition is satisfied if the amount of interdependence among the $\boldsymbol{Y}_t$ components is small or $\boldsymbol{c}_j(\tau_j)$ is small, such that the model is, indeed, driven by $\boldsymbol{X}_t$ only. Assumption 3(ii) determines that $\boldsymbol{\Lambda}(\boldsymbol{\tau})$ satisfies a monotonicity property for all possible values of $\boldsymbol{\Pi}(\boldsymbol{\tau}, \boldsymbol{x})$.

## 5. Asymptotic properties

For each element of $\boldsymbol{Y}_t$, i.e., $Y_{jt}$ with $j \in \{1, \ldots, m\}$, and conditioning set $\boldsymbol{W}_t^j = \left(\boldsymbol{Y}_{-jt}^\top, \boldsymbol{X}_t^\top\right)^\top$ defined above, let $F_{jt}$ and $f_{jt}$ be the distribution and density functions, respectively, of $Y_{jt}$ conditional on $\boldsymbol{W}_t^j$. Then, we can write the $\tau$th conditional quantile

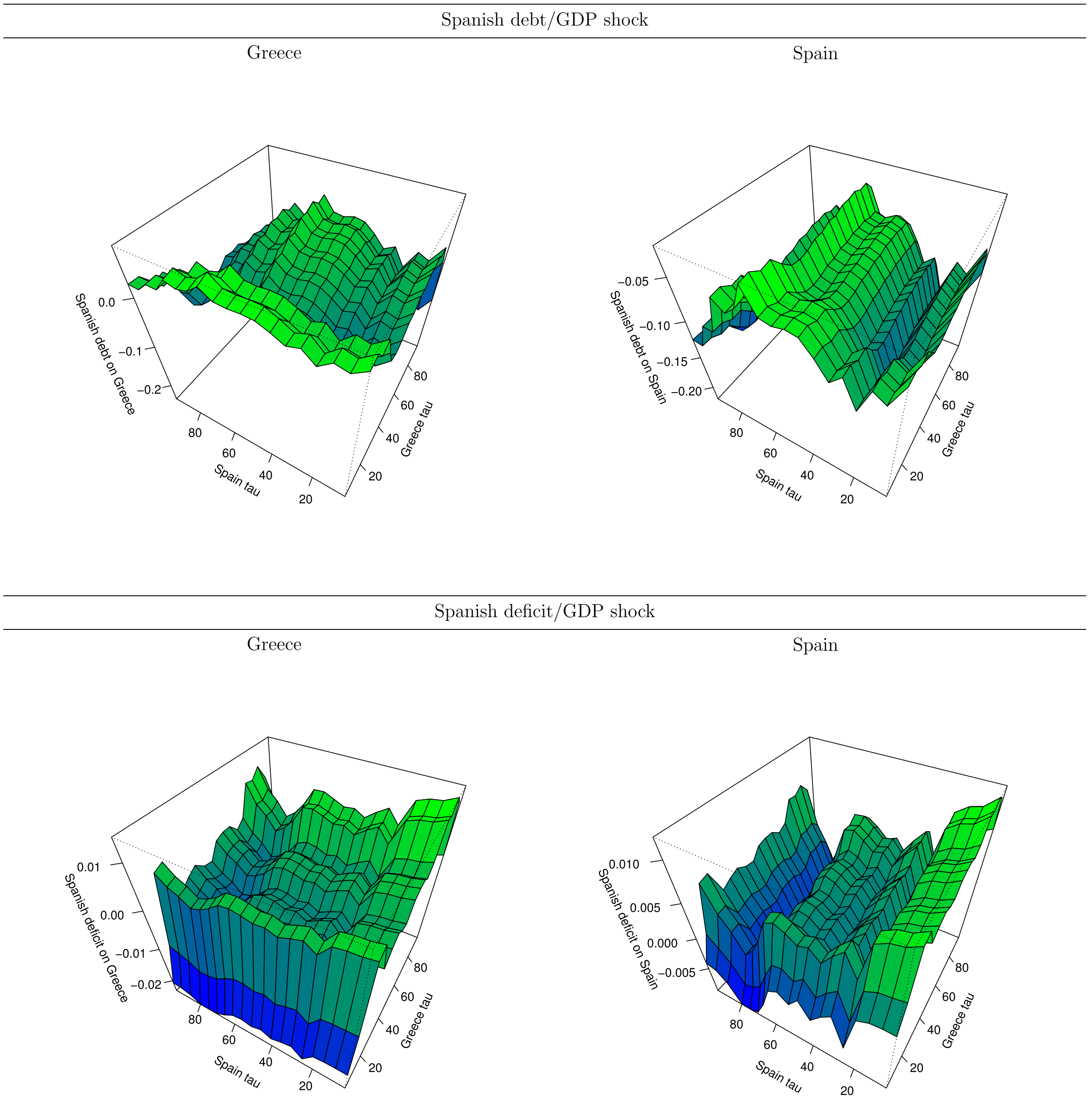


**Fig. 4.** Spanish debt/GDP and deficit/GDP effects on 10-year CDS, VDQ coefficients.

function of $Y_{jt}$, conditional on $\boldsymbol{W}_t^j$, as

$$Q_{Y_{jt}|\boldsymbol{W}_t^j}(\tau|\boldsymbol{W}_t^j) = \boldsymbol{c}_j(\tau)^\top \boldsymbol{Y}_{-jt} + \boldsymbol{b}_j(\tau)^\top \boldsymbol{X}_t + a_j(\tau). \tag{8}$$

Galvao et al. [10,11] consider the estimation of $\boldsymbol{\theta}_j(\tau) = (\boldsymbol{c}_j(\tau)^\top, \boldsymbol{b}_j(\tau)^\top, a_j(\tau))^\top$ and the regularity conditions to achieve uniform consistency and asymptotic normality in time-series models.

For a given sample $\{(\boldsymbol{y}_t, \boldsymbol{x}_t)\}_{t=1}^T$, let the QR estimator be

$$\hat{\boldsymbol{\theta}}_j(\tau) = \{\hat{\boldsymbol{c}}_j(\tau)^\top, \hat{\boldsymbol{b}}_j(\tau)^\top, \hat{a}_j(\tau)\}^\top = \underset{(\boldsymbol{c}^\top, \boldsymbol{b}^\top, a) \in \mathbb{R}^{k+m}}{\arg\min} \sum_{t=1}^T \{\rho_\tau(y_{jt} - \boldsymbol{c}^\top \boldsymbol{y}_{-jt} - \boldsymbol{b}^\top \boldsymbol{x}_t - a)\}.$$

The following assumptions provide sufficient conditions for consistency and asymptotic normality of $\hat{\boldsymbol{\theta}}_j(\tau)$ for all $\tau \in (0, 1)$ and extend the directional quantile model of [14] to the dependent time-series framework.

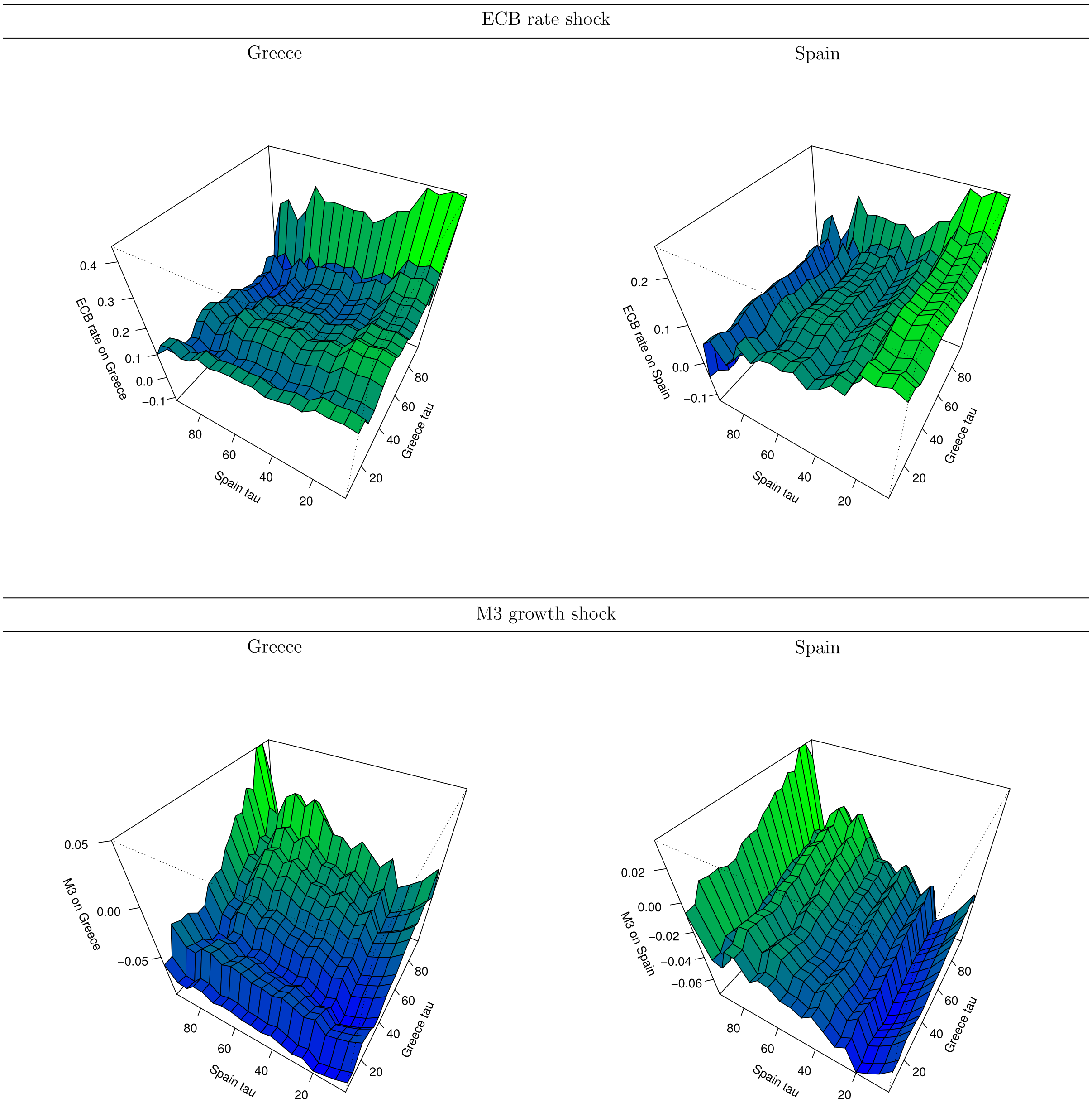


**Fig. 5.** ECB rate and M3 growth effects on 10-year CDS, VDQ coefficients.

**Assumption 4.**

(i) The sequence $(\boldsymbol{Y}_t, \boldsymbol{X}_t)_{t=1}^{T}$ is strictly stationary, ergodic, and $\rho$-mixing, with $\rho$-mixing coefficients' satisfying $\sum_{i=1}^{\infty} \rho_i^{1/2} < \infty$, with distribution $F$ that is absolutely continuous with respect to the Lebesgue measure with $\text{supp}(F) \subset (\mathcal{Y}, \mathcal{X}) \subset \mathbb{R}^{m+k}$, where $(\mathcal{Y}, \mathcal{X})$ is compact. Further, we assume that $F$ admits a density function that is continuous and non-zero over $(\mathcal{Y}, \mathcal{X})$.

(ii) $\text{E}(\|(\boldsymbol{Y}_t, \boldsymbol{X}_t)\|^{2+\epsilon}) < \infty$ with $\epsilon > 0$, $\|\cdot\|$ the Euclidean norm, and $\max_{(\boldsymbol{Y}_t, \boldsymbol{X}_t)} \|(\boldsymbol{Y}_t, \boldsymbol{X}_t)\| = O(\sqrt{T})$.

**Assumption 5.** For all $\tau \in \mathcal{T}$ and $j \in \{1, \ldots, m\}$, $\mathcal{T}$ a compact set in $(0, 1)$, the following hold true.

(i) There exists a unique

$$\boldsymbol{\theta}_j(\tau) = (\boldsymbol{c}_j(\tau)^\top, \boldsymbol{b}_j(\tau)^\top, a_j(\tau))^\top = \mathop{\arg\min}_{(\boldsymbol{c}^\top, \boldsymbol{b}^\top, a) \in \mathbb{R}^{k+m}} \text{E}\{\rho_\tau(Y_{jt} - \boldsymbol{c}^\top \boldsymbol{Y}_{-jt} - \boldsymbol{b}^\top \boldsymbol{X}_t - a)\} \in \Theta,$$

where $\Theta$ is a compact set in $\mathbb{R}^{k+m}$.

(ii) Let $\Omega_{j0} = \mathrm{E}(\boldsymbol{W}_t^j \boldsymbol{W}_t^{j\top})$ and $\Omega_{j1} = \mathrm{E}[f_{jt}\{F_{jt}^{-1}(\tau)\}\boldsymbol{W}_t^j \boldsymbol{W}_t^{j\top}]$. Then the matrix $\Sigma_j(\tau) = \Omega_{j1}(\tau)^{-1}\Omega_{j0}\Omega_{j1}(\tau)^{-1}$ exists and is non-singular.

The following lemma summarizes the asymptotic distribution of the univariate directional quantile coefficients, $\hat{\boldsymbol{\theta}}_j(\tau)$ for all $\tau \in (0, 1)$.

**Lemma 1.** *Under Assumptions 4 and 5, the limiting distribution of the QR estimators satisfies for $\tau \in \mathcal{T}$, $\mathcal{T}$ a compact set in $(0, 1)$, $j \in \{1, \ldots, m\}$,*

$$\Sigma_j(\tau)^{-1/2}\sqrt{T}\,\{\hat{\boldsymbol{\theta}}_j(\tau) - \boldsymbol{\theta}_j(\tau)\} \rightsquigarrow B_{k+m}(\tau),$$

*as $T \to \infty$, where $B_{k+m}(\tau)$ represents a $(k + m)$-dimensional Brownian Bridge.*

**Proof.** The result follows from an application of the quantile autoregressive model of [10]. □

The VDQ model is constructed after all the directional quantiles are estimated. In particular, $(\hat{\boldsymbol{B}}(\boldsymbol{\tau}), \hat{\boldsymbol{A}}(\boldsymbol{\tau}))$ is a function of $\{\hat{\boldsymbol{\theta}}_j(\tau_j)\}_{j=1}^m$, as given by Definition 1. Assumption 1 guarantees that the continuous mapping theorem can be applied to obtain consistent estimates of the VDQ model parameters.

## 6. Empirical application

We apply the VDQ estimator to model sovereign bonds interdependence among European countries. A credit default swap (CDS) is a contract in which the buyer of the CDS makes a series of premium payments to the seller and, in exchange, receives a payoff if a bond (or contract) goes into default. A CDS is a direct measure of the default risk but not of the probability of default, as the price of a CDS depends both on the probability of default and on the expected recovery value of the defaulted bond. The standard specification adopted for sovereign bonds assumes a persistent process that reverts toward a time-varying mean determined by country-specific factors, namely, fiscal and growth fundamentals, and common factors, measuring market appetite for risk; see [8].

Favero [8] proposes a global VAR specification [21,22] to model CDS interdependence. Let $Y_{jt}$ be the natural logarithm of the CDS index for country $j$, and let $Y_{jt} - Y_{\mathrm{GER}t}$ denote the CDS spread with respect to Germany's CDS. Germany (GER) was perceived as a "safe-haven" in international financial markets after the 2008–2009 financial crisis [2]. Following Favero [8], we consider a model for $\Delta(Y_{jt} - Y_{\mathrm{GER}t})$ as a function of $Y_{jt-1} - Y_{\mathrm{GER}t-1}$ and a set of covariates $\boldsymbol{X}_t$. These covariates are given by fiscal and growth fundamentals denoted by $b_{jt}$, debt/GDP (in %), and $d_{jt}$, deficit/GDP (in %), respectively, together with $(Baa_{t-1} - Aaa_{t-1})$, a global risk aversion measure.

We also consider two monetary variables of interest, namely, the Euro-M3 annual growth rate (in %), $m3_t$, and the European Central Bank's (ECB) refinancing rate (in %), $ecb_t$, to evaluate the effect of aggregate European monetary shocks.

Data on daily CDS with maturities between 1 and 10 years are provided by Bloomberg and S&P Capital-IQ from 2006 to 2014. In particular, we consider the monthly mean of 10-years CDS bonds. Data on European countries, Euro-M3 annual growth rate, and the ECB rate are obtained from Eurostat. Finally, the Baa-Aaa spread variable is a US corporate long-term risk-aversion measure, computed on the basis of the data made available in the FRED database of the Federal Reserve of St. Louis.

In terms of the VDQ model, for each country $j$, the $\tau_j$ directional quantile shows the CDS QR model conditional on other $-j$ countries' performance and $\boldsymbol{X}_t$. The VDQ model allows for different quantile configurations of all countries taken together. Thus, for instance, a certain covariate of interest may have a different effect on each country CDS level, depending on its specific quantile (i.e., $\tau_j$) and on other countries' quantiles (i.e., $\boldsymbol{\tau}_{-j}$). In our case, some European countries may be key predictors of other countries' market valuation, and, as such, parameter heterogeneity needs to be analyzed using its own and other countries' quantile indexes together.

We estimate a bivariate VDQ model for Greece and Spain. Greece has been the most seriously damaged country after the European debt crisis (with at least three bailouts). Spain has been affected mainly by contagion because it started with a supposedly appropriate value of fundamentals. We thus use the VDQ model to evaluate the potential interdependence of these two seemingly unrelated countries that share their dependence on the Euro and the rule of the European institutions. We report the VDQ coefficients on the $(\tau_{\mathrm{Greece}}, \tau_{\mathrm{Spain}})$ plane, and, in particular, we use the grid $\tau_{\mathrm{Greece}} \in \{0.05, \ldots, 0.95\}$ and $\tau_{\mathrm{Spain}} \in \{0.05, \ldots, 0.95\}$.

Fig. 3 shows the effect of changes in Greece's debt and deficit ratios to GDP on Greek and Spanish CDSs. The Greek debt ratio affects both Greece and Spain, and its effect is monotonic on both $\tau_{\mathrm{Greece}}$ and $\tau_{\mathrm{Spain}}$ for both countries. The Greek deficit ratio, however, shows heterogeneity only in the $\tau_{\mathrm{Greece}}$-axis direction but not in the $\tau_{\mathrm{Spain}}$-axis direction. We can thus interpret this result as both CDSs' being affected by the Greek debt performance, although Greek deficits are important only for the Greek CDS.

Fig. 4 presents an analysis of changes in the Spanish debt and deficit ratios to GDP on Greek and Spanish CDSs. The Spanish debt ratio has a negative effect on Spain's CDS and a heterogeneous and weaker (as compared to the reverse analysis) effect on Greece's CDS. The Spanish deficit has a symmetric effect on both countries.

Finally, Fig. 5 presents the effect of monetary changes in the Euro area on CDS performance. A rise in the ECB rate and a decrease in the M3 growth correspond to a tightening of the monetary policy. The results show that a tightening of the aggregate monetary conditions in the Euro area has a heterogeneous effect on Greece and Spain. In Greece, the ECB rate effect depends on both countries' conditional quantiles, for which the largest effect corresponds to high $\tau_{\text{Greece}}$ and low $\tau_{\text{Spain}}$. In Spain, however, the effect is clearly monotonic, decreasing on Spain's $\tau$, and only weakly monotonic on Greece's $\tau$. Similar patterns are observed for the effect of M3 growth.

## 7. Conclusion

In this paper, we propose a model for analyzing reduced form heterogeneity in a multivariate time-series context. Based on directional quantiles, we propose to solve for a fixed point on the multivariate quantile space. The result is a generalization of the vector autoregressive model for the conditional mean to conditional multivariate quantiles.

The present paper can be extended in several directions. First, we have addressed only linear QR models for each separate direction, and, as such, the VDQ model requires a simple matrix inversion. This model could be applied to nonlinear and nonparametric models for each direction, and the VDQ model would thus be a fixed-point solution to a nonlinear system of equations.

Second, the model should be further evaluated in terms of in-sample and out-sample dynamic forecasting. In particular, given fixed covariates for which we would like to forecast, random draws on the $m$-dimensional unit ball should be able to forecast the $m$-dimensional density function. As QR provides a flexible model to construct univariate density estimations, the VDQ model could be applied to multivariate density frameworks.

Third, the VDQ model could be used for building heterogeneous impulse-response functions. These would be quite useful for macroeconomics models, where complex models are analyzed in terms of specific shocks.

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## References

[1] C. Bernard, C. Czado, Conditional quantiles and tail dependence, J. Multivariate Anal. 138 (2015) 104–126.
[2] K. Bernoth, J. von Hagen, L. Schuknecht, Sovereign risk premiums in the European government bond market, J. Int. Money Financ. 31 (2012) 975–995.
[3] G. Carlier, V. Chernozhukov, A. Galichon, Vector quantile regression, Ann. Statist. 44 (2016) 1165–1192.
[4] P. Chaudhuri, On a geometric notion of quantiles for multivariate data, J. Amer. Statist. Assoc. 91 (1996) 862–872.
[5] V. Chernozhukov, A. Galichon, M. Hallin, M. Henry, Monge–Kantorovich Depth, Ranks, Quantiles, and Signs, CEMMAP Working Paper CWP04/15, 2015.
[6] A. Chesher, Identification in nonseparable models, Econometrica 71 (2003) 1405–1441.
[7] R.F. Engle, S. Manganelli, CAViaR: conditional autoregressive value at risk by regression quantiles, J. Bus. Econom. Statist. 22 (2004) 367–381.
[8] C.A. Favero, Modeling and forecasting government bond spreads in the euro area: a GVAR model, J. Econometrics 177 (2013) 343–356.
[9] R. Fraiman, B. Pateiro-López, Quantiles for finite and infinite dimensional data, J. Multivariate Anal. 108 (2012) 1–14.
[10] A.F. Galvao, G. Montes-Rojas, J. Olmo, Threshold quantile autoregressive models, J. Time Series Anal. 32 (2011) 253–267.
[11] A.F. Galvao, G. Montes-Rojas, S. Park, Quantile autoregressive distributed lag model with an application to house price returns, Oxford Bull. Econom. Statist. 75 (2013) 307–321.
[12] C. Genest, A.-C. Favre, Everything you always wanted to know about copula modeling but were afraid to ask, J. Hydrologic Eng. 12 (2007) 347–368.
[13] C. Genest, J. Nešlehová, Copulas and copula models, in: A.H. El-Shaarawi, W.W. Piegorsch (Eds.), in: Encyclopedia of Environmetrics, vol. 2, Wiley, Chichester, 2012, pp. 541–553.
[14] M. Hallin, D. Paindaveine, M. Šiman, Multivariate quantiles and multiple-output regression quantiles: from $\ell_1$ optimization to halfspace depth, Ann. Statist. 38 (2010) 635–669.
[15] H. Han, O. Linton, T. Oka, Y.-J. Whang, The cross-quantilogram: measuring quantile dependence and testing directional predictability between time series, J. Econometrics 193 (2016) 251–270.
[16] R. Koenker, Z. Xiao, Quantile autoregression, J. Amer. Statist. Assoc. 101 (2006) 980–990.
[17] V. Koltchinskii, M-estimation, convexity and quantiles, Ann. Statist. 25 (1997) 435–477.
[18] L. Ma, R. Koenker, Quantile regression methods for recursive structural equation models, J. Econometrics 134 (2006) 471–506.
[19] D. Paindaveine, M. Šiman, On directional multiple-output quantile regression, J. Multivariate Anal. 102 (2011) 193–212.
[20] D. Paindaveine, M. Šiman, Computing multiple-output regression quantile regions, Comput. Statist. Data Anal. 56 (2012) 840–853.
[21] M. Pesaran, T. Schuerman, B.-J. Treutler, S. Wiener, Macroeconomic dynamics and credit risk: a global perspective, J. Money Credit Bank. 38 (2006) 1211–1261.
[22] M. Pesaran, T. Schuerman, S. Wiener, Modeling regional interdependencies using a global error-correcting macro-econometric model, J. Bus. Econom. Statist. 22 (2004) 129–162.
[23] Y. Wei, An approach to multivariate covariate-dependent quantile contours with application to bivariate conditional growth charts, J. Amer. Statist. Assoc. 103 (2008) 397–409.
[24] H. White, T.-H. Kim, S. Manganelli, VAR for VaR: measuring tail dependence using multivariate regression quantiles, J. Econometrics 187 (2015) 169–188.