
Volume 104, No. 2, August 2009
ISSN 0165-1765
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Economics Letters 104 (2009) 66–68

Contents lists available at ScienceDirect

Economics Letters

journal homepage: www.elsevier.com/locate/ecolet

Testing under local misspecification and artificial regressions

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ARTICLE INFO

Article history:
Received 21 May 2008
Received in revised form 24 March 2009
Accepted 6 April 2009
Available online 15 April 2009

Keywords:
Specification tests
LM tests
Artificial regression

JEL classification:
C12

ABSTRACT

An additivity property of LM tests is derived, linking joint, marginal and Bera–Yoon "adjusted" tests, hence the latter can be derived as the difference of the first two. An artificial regression framework provides an intuitive geometrical illustration of the Bera–Yoon principle.

1. Introduction

The Lagrange Multiplier (LM) or Rao's score test is a popular choice when the estimation of a parametric "null" model is computationally convenient, and is extensively used in practice. Nevertheless Davidson and MacKinnon (1987) and Saikkonen (1989) showed that its performance may be negatively affected by the presence of misspecified alternatives, that is, when the true model does not correspond to the alternative hypothesis postulated by the researcher. Bera and Yoon (1993) proposed a modification of the LM test that is robust to locally misspecified alternatives, and that has been successfully implemented in many applications. For example, Bera et al. (2001) found that the presence of first order serial correlation makes the standard Breusch and Pagan (1980) test for random effects reject its null hypothesis too often, thus implying that rejections may be due to random effects but also to serial correlation. Based on the Bera–Yoon (BY henceforth) principle, they derived an adjusted LM test for random effects that is not affected by the presence of local serial correlation. In a similar fashion, Anselin et al. (1996) derived tests for spatial autocorrelation that are not sensitive to the presence of local spatial lag dependence. Baltagi and Li (1999) used the BY principle to obtain tests for functional form misspecification and spatial correlation.

This paper discusses a useful *additivity property* of *LM* tests, that links joint, marginal and BY "adjusted" tests. In particular, joint LM tests are shown to be the sum of the BY adjusted LM test for the parameters of interest and the marginal LM test for the nuisance parameters. This is a very useful result, since in many cases joint and marginal tests are easily available. Thus, the BY adjusted tests can be constructed by taking the appropriate differences. Bera et al. (2007) provide a pure algebraic proof of this result based on matrix algebra. We use an artificial regression framework that, besides leading to a very simple proof, provides an intuitive illustration of the relationship between joint, marginal and BY adjusted tests.

2. Joint, marginal and adjusted LM tests

Consider a simple parametric statistical model for an *i.i.d.* sample of size n, fully characterized by its log-likelihood function $l(\theta)=\sum_{i=1}^{n} l_i(\theta)$, where $\theta\in\Re^p$, and $l_i(\theta)$ is the log-likelihood for each observation. Let $d(\theta)\equiv\partial l(\theta)/\partial\theta'$ be the score vector, and $J(\theta)$ the information matrix for a single observation. The parameter vector will be partitioned as $\theta=(\psi',\phi')'$, where ψ and ϕ are vectors in open subsets of $\Re^{p_1}$ and $\Re^{p_2}$, respectively, so $p=p_1+p_2$. Let $d_\psi(\theta)$ and $d_\phi(\theta)$ denote the score vectors corresponding to each of these parameters, and partition the information matrix accordingly as

$$J(\theta)=\begin{bmatrix} J_{\psi\psi}(\theta) & J_{\psi\phi}(\theta) \\ J_{\phi\psi}(\theta) & J_{\phi\phi}(\theta)\end{bmatrix}.$$

The situation of interest for this paper is when a researcher is interested in evaluating the null hypothesis H_0^{ψ}: $\psi=\psi_0$ and believes that $\phi=\phi_0$. To simplify notation we will first consider the case when

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0165-1765/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.econlet.2009.04.005

no parameters are estimated, and later on we will provide a generalization to the case of estimated parameters. The LM statistic for H_0^ψ, which we will refer to as a "marginal" test, is

$$LM_\psi = \frac{1}{n} d_\psi(\theta_0)' J_{\psi\psi}^{-1}(\theta_0) d_\psi(\theta_0), \tag{1}$$

where $\theta_0 = (\psi'_0, \phi'_0)'$, and a standard result is that under the null hypothesis, LM_ψ has an asymptotic *central* chi-squared distribution with p_1 degrees of freedom.

Davidson and MacKinnon (1987) and Saikkonen (1989) showed that when $\phi \neq \phi_0$, that is, when the alternative is misspecified, LM_ψ no longer has a *central* chi-squared distribution. In particular, they show that when H_0^ψ holds but $\phi = \phi_0 + \delta/\sqrt{n}$, with $0 \leq \delta < \infty$, $LM_\psi \xrightarrow{d} \chi^2_{p_1}(\lambda)$, where the *non-centrality* parameter λ is given by $\lambda = \delta' J_{\phi\psi} J_{\psi\psi}^{-1} J_{\psi\phi} \delta$.

Using the above result Bera and Yoon (1993) proposed the following adjusted version of the LM statistic,

$$LM^*_\psi = \frac{1}{n} d^*_\psi(\theta_0)' J^{*-1}_{\psi\psi}(\theta_0) d^*_\psi(\theta_0), \tag{2}$$

where $d^*_\psi(\theta_0) \equiv d_\psi(\theta_0) - J_{\psi\phi}(\theta_0) J_{\phi\phi}^{-1}(\theta_0) d_\phi(\theta_0)$ and $J^*_{\psi\psi}(\theta_0) \equiv J_{\psi\psi}(\theta_0) - J_{\psi\phi}(\theta_0)\ J_{\phi\phi}^{-1}(\theta_0) J_{\phi\psi}(\theta_0)$, and showed that when H_0^ψ holds but when $\phi = \phi_0 + \delta/\sqrt{n}$, LM^*_ψ has asymptotically a *central* chi-squared distribution with p_1 degrees of freedom. That is, the LM^*_ψ test is insensitive to the local departure of ϕ away from ϕ_0, hence providing a valid test framework under local misspecification. We will refer to this test as the "adjusted" LM test.

Finally, consider the "joint" LM test for the null H_0: $\psi = \psi_0$, $\phi = \phi_0$:

$$LM_{\psi\phi} = \frac{1}{n} d(\theta_0)' J^{-1}(\theta_0) d(\theta_0), \tag{3}$$

which under H_0 has asymptotically a central chi-squared distribution with $p_1 + p_2$ degrees of freedom.

3. The additivity property and artificial regressions

The key result is the following.

Lemma 1. $LM_{\psi\phi} = LM^*_\psi + LM_\phi = LM_\psi + LM^*_\phi$.

This is an important result, since it implies that BY adjusted tests can be simply derived as the difference between joint and marginal tests. Bera et al. (2007) provide a pure algebraic proof of this result. Nevertheless, our analysis here is based on artificial regressions, which provide a useful asymptotically equivalent framework that helps to intuitively interpret the relationship among the three tests, and leads to a very simple geometric proof of the result.

It is well known that LM tests can be obtained from artificial regressions (see MacKinnon, 1992, for a detailed exposition of artificial regressions and LM tests). Consider H_0: $\psi = \psi_0$, $\phi = \phi_0$, and the corresponding test statistic given by (1). Consider the following artificial regression

$$\iota = G(\theta)c + u, \tag{4}$$

where ι is an $n \times 1$ vector of ones, $G(\theta)$ is an $n \times p$ matrix with typical element

$$G(\theta)_{ik} = \frac{\partial l_i(\theta)}{\partial \theta_k}, \quad i = 1, \dots, n, \quad k = 1, \dots, p,$$

and u is a residual vector. Note that $d(\theta) = G(\theta)'\iota$. The outer-product-of-gradients (OPG) artificial regression relies on the fact that $n^{-1}G'G = J + o_p(1)$, so the explained sum of squares (ESS) of regression model (4) satisfies the following asymptotic equality

$$\iota' G(G'G)^{-1} G'\iota = \frac{1}{n} d' J^{-1} d + o_p(1). \tag{5}$$

We will write the OPG approximation to the joint LM test as

$$LM^o_{\psi\phi} = \iota' G(\theta_0) \left[G(\theta_0)' G(\theta_0)\right]^{-1} G(\theta_0)'\iota.$$

In order to express the marginal tests in terms of artificial regressions let $G_\psi(\theta)$ and $G_\phi(\theta)$, respectively, denote the columns of $G(\theta)$ corresponding to ψ and ϕ, thus $G(\theta) = [G_\psi(\theta)\ G_\phi(\theta)]$, and $d_\psi(\theta) = G_\psi(\theta)'\iota$ and $d_\phi(\theta) = G_\phi(\theta)'\iota$. Asymptotically, the marginal test LM_ψ is simply the ESS from regressing ι on $G_\psi(\theta_0)$, and therefore, its OPG approximation is

$$LM^o_\psi = \iota' G_\psi(\theta_0)' \left[G_\psi(\theta_0)' G_\psi(\theta_0)\right]^{-1} G_\psi(\theta_0)'\iota.$$

Interestingly, the BY statistic can also be expressed in terms of the OPG regression, as the ESS of regressing ι on the residuals obtained from the multivariate regression of $G_\psi(\theta_0)$ on $G_\phi(\theta_0)$. In order to establish this result, let $G^{*\prime}_\psi \equiv G'_\psi - G'_\psi G_\phi (G'_\phi G_\phi)^{-1} G'_\phi$, where the dependence on θ_0 has been omitted for simplicity. The ESS of regressing ι on G^*_ψ is $\iota' G^*_\psi (G^{*\prime}_\psi G^*_\psi)^{-1} G^{*\prime}_\psi \iota$, and it is easy to check

$$\begin{aligned} \frac{1}{\sqrt{n}} G^{*\prime}_\psi \iota &= \frac{1}{\sqrt{n}} \left(G'_\psi \iota - G'_\psi G_\phi \left(G'_\phi G_\phi\right)^{-1} G'_\phi \iota\right) \\ &= \frac{1}{\sqrt{n}} \left(d_\psi - J_{\psi\phi} J_{\phi\phi}^{-1} d_\phi\right) + o_p(1), \end{aligned}$$

and $n^{-1} G^{*\prime}_\psi G^*_\psi = n^{-1}(G'_\psi G_\psi - G'_\psi G_\phi (G'_\phi G_\phi)^{-1} G'_\phi G_\psi) = J_{\psi\psi} - J_{\psi\phi} J_{\phi\phi}^{-1} J_{\phi\psi} + o_p(1)$. Therefore, we can write the OPG approximation of LM^*_ψ as

$$LM^{*o}_\psi = \iota' G^*_\psi(\theta_0) \left(G^*_\psi(\theta_0) G^*_\psi(\theta_0)\right)^{-1} G^*_\psi(\theta_0)'\iota.$$

Consequently, the adjusted LM test can be seen as a simple LM test where in a first stage the effect of ϕ is removed by orthogonal projection, i.e., considering the residuals from the regression of G_ψ on G_ϕ.

The asymptotically equivalent of Lemma 1 in the artificial regressions framework is:

Lemma 2. $LM^o_{\psi\phi} = LM^{o*}_\psi + LM^o_\phi = LM^o_\psi + LM^{o*}_\phi$.

We use the artificial regression framework to provide a simple and intuitive geometrical illustration of this result. Consider Fig. 1. Three vectors are represented with solid arrows. ι represents the vector of ones, and G_ϕ and G_ψ represent the vectors of scores. The joint test $LM^o_{\psi\phi}$ is represented by the segment $\overline{ac}$. Its Euclidean length is the ESS of projecting ι on the space spanned by G_ϕ and G_ψ, as established by the artificial regression result. The marginal test LM^o_ϕ is represented by the segment $\overline{ad}$, whose length is the ESS of projecting ι on the space spanned

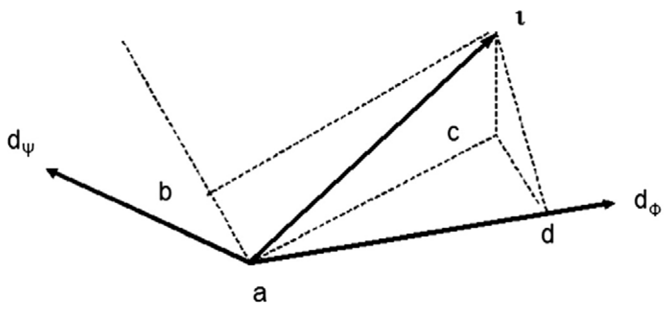

Fig. 1. Geometrical representation of joint, marginal and BY adjusted tests.

by G_ϕ. Finally, the adjusted LM test $LM_\psi^{o^*}$ is represented by the segment $\overline{ab}$. The line containing $\overline{ab}$ is the orthogonal complement of the space spanned by G_ϕ, so the residuals of projecting G_ψ on G_ϕ must lie along this ray. Hence $\overline{ab}$ is the ESS of projecting ι on this space.

Based on this picture it easy to see that $\overline{ac}$ (representing $LM_{\psi\phi}^o$) can be constructed as the sum of the orthogonal vectors $\overline{ab}\left(=\overline{dc}\right)$ and $\overline{ad}$ that represent, respectively, $LM_\psi^{o^*}$ and LM_ϕ^o, and which is the desired result.

When G_ψ and G_ϕ are already orthogonal, there is no need for adjustments, and the joint test is simply the sum of the marginal tests LM_ψ^o and LM_ϕ^o, in which case the adjusted and unadjusted test statistics coincide.

A proof of Lemma 2 can be based on two standard results on OLS projections. The first one is that if in the linear model $Y=b_1X_1+b_2X_2+u$, X_1 and X_2 are orthogonal, then $ESS=ESS_1+ESS_2$, where ESS_j is the explained sum of squares of projecting Y on the space spanned by X_j, $j=1,2$. The second one is the regression

$$Y=a_1X_1+a_2\dot{X}_2+u_2,$$

where $\dot{X}_2 \equiv M_1X_2$, with $M_1 \equiv I-X_1(X_1'X_1)^{-1}X_1'$, that is, $\dot{X}_2$ are the residuals from running a regression of X_2 on X_1. Then the residuals of both regression models coincide, and hence their ESSs coincide too, since both models have the same explained variable. To establish Lemma 2, note that the joint test is asymptotically the ESS of regressing ι on G_ψ and G_ϕ so by our previous results, this ESS remains unaltered if ι is instead regressed on G_ψ^* and G_ϕ, because by construction G_ψ^* and G_ϕ are orthogonal. Finally, using the first OLS result, and the fact that $LM_\psi^{o^*}$ is the ESS of regressing ι on G_ψ^*, and LM_ϕ^o is the ESS of regressing ι on G_ϕ, we complete the proof.

In practice there will be unknown parameters that need to be estimated, say γ, such that the complete parameter vector can be decomposed as $\theta=(\gamma',\psi',\phi')'$. Let $\hat{\gamma}$ be the MLE of γ, under the joint null H_0: $\psi=\psi_0$, $\phi=\phi_0$, and let $\hat{\theta}=(\hat{\gamma}',\psi_0',\phi_0')'$. The marginal LM test for H_0: $\psi=\psi_0$ assuming $\phi=\phi_0$ becomes:

$$LM_{\psi\cdot\gamma}=\frac{1}{n}d_\psi\left(\hat{\theta}\right)'J_{\psi\cdot\gamma}^{-1}\left(\hat{\theta}\right)d_\psi\left(\hat{\theta}\right),$$

where $J_{\psi\cdot\gamma} \equiv J_\psi-J_{\psi\gamma}J_{\gamma\gamma}^{-1}J_{\gamma\psi}$. The joint LM test for H_0: $\psi=\psi_0$, $\phi=\phi_0$ is

$$LM_{\psi\phi\cdot\gamma}=\frac{1}{n}d_{\psi\phi}\left(\hat{\theta}\right)J_{\psi\phi\cdot\gamma}^{-1}\left(\hat{\theta}\right)d_{\psi\phi}\left(\hat{\theta}\right),$$

where $J_{\psi\phi\cdot\gamma} \equiv J_{\psi\phi}-J_{\psi\phi,\gamma}J_{\gamma\gamma}^{-1}J_{\gamma,\psi\phi}$, and, finally, the adjusted LM test is

$$LM_{\psi\cdot\gamma}^*=\frac{1}{n}d_\psi^{*\prime}\left(\hat{\theta}\right)J_{\psi\cdot\gamma}^{*-1}\left(\hat{\theta}\right)d_\psi^*\left(\hat{\theta}\right),$$

where $d_\psi^*=d_\psi-J_{\psi\phi\cdot\gamma}J_{\phi\phi\cdot\gamma}^{-1}d_\phi$ and $J_{\psi\cdot\gamma}^*=J_{\psi\cdot\gamma}-J_{\psi\phi\cdot\gamma}J_{\phi\phi\cdot\gamma}^{-1}J_{\phi\psi\cdot\gamma}$.

A more general result is then:

Lemma 3. $LM_{\psi\phi\cdot\gamma}=LM_{\psi\cdot\gamma}^*+LM_{\phi\cdot\gamma}=LM_{\psi\cdot\gamma}+LM_{\phi\cdot\gamma}^*$.

Our approach can be extended to prove this result using an OPG-based asymptotically equivalent version of Lemma 3. First note that by standard Taylor expansions and by properties of MLE,

$$\frac{1}{\sqrt{n}}d_\psi\left(\hat{\theta}\right)=\frac{1}{\sqrt{n}}d_\psi(\theta_0)-J_{\psi\gamma}(\theta_0)\sqrt{n}\left(\hat{\gamma}-\gamma_0\right)+o_p(1)$$

and

$$\frac{1}{\sqrt{n}}d_\phi\left(\hat{\theta}\right)=\frac{1}{\sqrt{n}}d_\phi(\theta_0)-J_{\phi\gamma}(\theta_0)\sqrt{n}\left(\hat{\gamma}-\gamma_0\right)+o_p(1).$$

and $\sqrt{n}\left(\hat{\gamma}-\gamma_0\right)=J_{\gamma\gamma}^{-1}(\theta_0)\frac{1}{\sqrt{n}}d_\gamma(\theta_0)+o_p(1)$. Second, the OPG approximation implies

$$\frac{1}{\sqrt{n}}d_\psi\left(\hat{\theta}\right)=\frac{1}{\sqrt{n}}G_\psi(\theta_0)'\iota-\left(G_\psi(\theta_0)'G_\gamma(\theta_0)\right)\times\left(G_\gamma(\theta_0)'G_\gamma(\theta_0)\right)^{-1}\frac{1}{\sqrt{n}}G_\gamma(\theta_0)'\iota+o_p(1)$$

and

$$\frac{1}{\sqrt{n}}d_\phi\left(\hat{\theta}\right)=\frac{1}{\sqrt{n}}G_\phi(\theta_0)'\iota-\left(G_\phi(\theta_0)'G_\gamma(\theta_0)\right)\times\left(G_\gamma(\theta_0)'G_\gamma(\theta_0)\right)^{-1}\frac{1}{\sqrt{n}}G_\gamma(\theta_0)'\iota+o_p(1).$$

It is worth to notice that $G_\psi-(G_\psi'G_\gamma)(G_\gamma'G_\gamma)^{-1}G_\gamma$ and $G_\phi-(G_\phi'G_\gamma)(G_\gamma'G_\gamma)^{-1}G_\gamma$ are the residuals of an artificial regression of G_ψ and G_ϕ, respectively, on G_γ. That is, the modified score vectors are orthogonal to the influence of deviations in γ through G_γ. Finally, we can apply Lemma 2 to these orthogonal scores, and the proof concludes by noting that the OPG asymptotic aproximation consistently estimates the information matrix.

4. Concluding remarks

The additivity property discussed in this paper provides an interesting link between joint, marginal, and the adjusted LM tests proposed by Bera and Yoon (1993), where the latter, unlike standard LM tests, are insensitive to local misspecification. This is a very useful result, since in many practical instances joint and marginal tests are available, so Bera–Yoon robust test statistics can be computed quite easily. As an application, consider the recent paper by Baltagi et al. (2006) that derived marginal and joint tests to detect heteroscedasticity in an error components model. Their marginal tests for each component assume homoscedasticity in the remaining component. Using our additivity result, an adjusted test for homoscedasticity in one error component that is insensitive to the local presence of heteroscedasticity in the other component can be obtained by taking the difference of the joint and a marginal test. As a relevant extension, Bera et al. (2008) recently derived adjusted tests under a more general estimation approach, such as the generalized method of moments (GMM). An interesting exercise for further research would be to generalize the additivity property for the MLE based tests to the GMM framework.

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