Econometrics **2015**, *3*, 654-666; doi:10.3390/econometrics3030654

OPEN ACCESS

econometrics

ISSN 2225-1146

www.mdpi.com/journal/econometrics

Article

On Bootstrap Inference for Quantile Regression Panel Data: A Monte Carlo Study

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Academic Editor: Kerry Patterson

Received: 26 June 2015 / Accepted: 19 August 2015 / Published: 10 September 2015

Abstract: This paper evaluates bootstrap inference methods for quantile regression panel data models. We propose to construct confidence intervals for the parameters of interest using percentile bootstrap with pairwise resampling. We study three different bootstrapping procedures. First, the bootstrap samples are constructed by resampling only from cross-sectional units with replacement. Second, the temporal resampling is performed from the time series. Finally, a more general resampling scheme, which considers sampling from both the cross-sectional and temporal dimensions, is introduced. The bootstrap algorithms are computationally attractive and easy to use in practice. We evaluate the performance of the bootstrap confidence interval by means of Monte Carlo simulations. The results show that the bootstrap methods have good finite sample performance for both location and location-scale models.

Keywords: quantile regression; bootstrap; fixed effects

JEL classifications: C13, C21, C23

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1. Introduction

Quantile regression (QR) for panel data has attracted interest in both theoretical and applied literature. It allows researchers to explore a range of conditional quantile functions, thereby exposing a variety of forms of conditional heterogeneity, and to control for unobserved individual characteristics. Controlling for individual-specific heterogeneity via fixed-effects (FE), while exploring heterogeneous covariate effects within the quantile regression framework, offers a more flexible approach to the analysis of panel data than that afforded by the classical Gaussian fixed and random effects estimators. The authors in [1,2] develop the asymptotic properties of these estimators (for other recent developments, see, e.g., [3–5]). In particular, the asymptotic variance of QR estimators depends on the density of the innovation term, and it is not easy to compute in practice. Thus, we argue that inference procedures and confidence interval construction can be greatly simplified by using bootstrap methods, and this paper evaluates bootstrapping procedures for panel quantile regression estimators with FE.

Bootstrapping techniques have been used to construct confidence intervals for QR in the cross-sectional context extensively. Buchinsky [6] uses Monte Carlo simulation to study several estimation procedures of the asymptotic covariance matrix in quantile regression models, and the results favor the bootstrap design. Hahn [7] shows that the construction of confidence intervals based for the QR estimators can be greatly simplified by using bootstrapping. Moreover, the confidence intervals constructed by the bootstrap percentile method have asymptotically correct coverage probabilities. Horowitz [8] proposes bootstrap methods for median regression models. Feng, He, and Hu [9] proposes an adaptation wild bootstrap methods for QR. Wang and He [10] develops inference procedures based on rank-score tests with random effects. In the panel data FE context, [11,12] use bootstrapping for constructing confidence intervals in the QR panel data. However, they do not provide evidence that this procedure is valid, nor do they provide an explicit methodology to implement the bootstrapping for panel data QR. This paper fills this gap.

Inference in panel datasets using least squares methods has been mainly used in asymptotic approximations for the construction of test statistics and the variances of the estimators. The use of bootstrap as an alternative to such asymptotic approximations has been considered, but its properties have not received the same amount of attention as in the cross-section or time series literature. The consideration of the bootstrap for panel data has focused on resampling in the time dimension, extending the work on the bootstrap in time series. Resampling in the cross-sectional dimension has received attention, as well. However, the literature on the combination of the two resampling schemes in panel data is very limited. Kapetanios [13] discusses bootstrapping for panel data when resampling occurs in both cross-sectional and time series dimensions. A treatment of resampling methods when N is large, but T is assumed small and fixed can be found in [14].

In this paper, we suggest inference procedures to construct confidence intervals for the parameters of interest in QR FE panel data models based on the application of bootstrap percentiles. This alternative approach circumvents the problem of estimating the sparsity function and the asymptotic covariance matrix. We apply the pairwise bootstrap resampling technique, since bootstrapping pairs is, in general, less sensitive to certain regularity conditions than bootstrapping residuals. Following [13], we study three different possibilities of sampling schemes for panel data QR: cross-sectional, temporal and cross-sectional and temporal. In the first case, we construct the bootstrap samples by resampling only from cross-sectional units with replacement. In the second case, the temporal resampling is only from the time series dimension, without changing the cross-sectional dimension. In the final case, a more general resampling scheme is proposed, which considers sampling from both cross-sectional and temporal dimensions.

A Monte Carlo study is conducted to study and evaluate the finite sample properties of the proposed inference procedure. We use the FE QR estimator. We evaluate the bootstrapping procedure using two different underlying models: (i) random-effects (where the covariates are not related to the individual effects); and (ii) fixed-effects (where covariates are correlated with the individual effects) models. We compute empirical rejection rates of the confidence interval for all different bootstrapping procedures. The results show that for the cross-sectional bootstrapping, the confidence intervals' empirical rejection rates are close to the nominal size, and more importantly, as expected, the results indicate that the empirical sizes approximate the nominal size as the sample size increases. Overall, the results show good performance for large N, with marginal improvements as T increases. In addition, similar results are obtained for the random- and fixed-effects models.

The rest of the paper is organized as follows. Section 2 briefly presents the fixed effects quantile regression model. Section 3 describes the bootstrap inference procedures. Section 4 describes the Monte Carlo experiment and results. Conclusions appear in Section 5.

2. The Model and Estimator

2.1. The Model

This paper considers a QR model with individual FE. Thus, the τ-th conditional quantile function of the response y_{it} of the t-th observation on the i-th individual can be represented as

$$Q_{y_{it}}(\tau|x_{it}, \eta_i) = \eta_i(\tau) + x_{it}'\beta(\tau), \tag{1}$$

where y_{it} is the response variable, η_i denotes the individual FE, x_{it} is a p-vector of exogenous covariates and $Q_{y_{it}}(\tau|x_{it}, \eta_i)$ is the conditional τ-quantile of y_{it} given (x_{it}, η_i). In Model (1), both the effects of the covariates (x_{it}) and the individual-specific effects are allowed to depend on the quantile, τ, of interest. In general, each η_i and β can depend on τ, but we assume τ to be fixed throughout the paper and suppress such a dependence for notational simplicity. The η's are intended to capture some individual specific source of variability, or "unobserved heterogeneity," that was not adequately controlled for by other covariates.

It is important to note that in the QR framework there is no transformation that can eliminate the FE; thus, we are required to deal with the full problem. This intrinsic difficulty has been recognized by [11], among others, and is clarified by Koenker and Hallock [15] (p.19) : "Quantiles of convolutions of random variables are rather intractable objects, and preliminary differencing strategies familiar from

Gaussian models have sometimes unanticipated effects." Therefore, following [2], we consider the FE estimation of β, which is implemented by treating each individual effect as a parameter to be estimated.[1]

2.2. Fixed-Effects Quantile Regression Estimator

We consider the standard QR FE estimation, which is implemented by treating each individual effect as a parameter to be estimated. The estimator is defined by

$$(\hat{\eta}, \hat{\beta}) = \arg\min_{\eta,\beta} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau(y_{it} - z'_{it}\eta - x'_{it}\beta), \quad (2)$$

where z_{it} identifies the individual FE, $\eta = (\eta_1, ..., \eta_N)'$ is the $N \times 1$ vector of individual specific effects or intercepts, β is a vector of slope parameters and $\rho_\tau(u) := u(\tau - I(u < 0))$ as in [16].

The asymptotic properties of the $(\hat{\eta}, \hat{\beta})$ estimators are derived in [2]. They show consistency and asymptotic normality of the estimator under a condition on the sample size growth. This assumption is relatively stringent and basically requires the time dimension to grow quickly relative to the cross-section, *i.e.*, $N^2/T \to 0$. However, a detailed discussion on such assumptions is outside the scope of this paper.[2]

3. Inference

3.1. The Bootstrap

The main concern of this paper is the application of bootstrapping procedures to the problem of constructing confidence intervals for the slope coefficients of the QR FE panel data model in (1), based on the point estimation methods described in the previous section.

The bootstrap resampling methods are designed to be used when the innovations of a regression are not identically distributed. The bootstrapping procedures are based on the so-called (y, x)-pairs bootstrap, which was originally proposed by [17]. This is a fully-nonparametric procedure that is applicable to a wide variety of models. Unlike resampling residuals, the pairs' bootstrap is not limited to iid errors in regression models. In the pairs' bootstrap, instead of resampling the dependent variable, or

1 An alternative model was developed by [1], where the individual effects are the same across quantiles. In most applications, the time series dimension T is relatively small compared to the number of individuals N. Therefore, it might be difficult to estimate a τ-dependent distributional individual effect. The restriction of the individual effects, η, to be independent of the specific quantile, τ, is implemented by estimating the model for several quantiles simultaneously.

2 The work in [1] introduced a general approach to estimate quantile regression models for panel data with FE that may be subject to shrinkage by l_1 regularization methods. It is well know that the optimal estimator for the random effects Gaussian model involves shrinking the individual effects toward a common value. When there is an intercept in the model, this common value can be taken to be the conditional central tendency of the response at a point determined by the centering of the other covariates. In the quantile regression model, this would be some corresponding conditional quantile of the response. Particularly, when N is large relative to T, shrinkage may be advantageous in controlling the variability introduced by the large number of estimated individual-specific parameters.

residuals, possibly centered or rescaled, it bootstraps pairs consisting of an observation of the dependent variable along with the vector of explanatory variables for that same observation.

However, this bootstrap implicitly assumes that the pairs (y_{it}, x_{it}) are independent. Although this is still a restrictive assumption, ruling out any form of dependence among observations, it does allow for arbitrary forms of heteroskedasticity of y_{it} conditional on x_{it}. The objects resampled are iid drawings from the joint distribution of y_{it} and x_{it}. Each bootstrap sample consists of some of the original pairs once, some of them more than once and some of them not at all. This procedure does not condition on x and does not assume that the innovation term is iid.

As in [13], we discuss several possibilities for bootstrap resampling schemes that can be applied in panel datasets, such as the cross-sectional resampling, the temporal resampling and the cross-sectional and temporal resampling.

Cross-sectional resampling: The first scheme is the cross-sectional resampling, which consists of resampling Y and X with replacement from the cross-section dimension with probability $1/N$, maintaining intact the temporal structure for each individual i. Thus, in this case, let $Y^* = (y_{i_1}, ..., y_{i_s}, ..., y_{i_N})$, where each element of the vector of indices $(i_1, ..., i_N)$ is obtained by drawing with replacement from $(1, ..., N)$, and each element is $y_{i_s} = (y_{i_s1}, ..., y_{i_sT})$ for $s \in (i_1, ..., i_N)$. The same vector of indices is used to obtain X^*.

Temporal resampling: The second scheme consist of resampling both Y and X with replacement from the temporal dimension for each individual with probability $1/T$, maintaining fixed the individual structure. The implementation constructs, for each $i \in (1, ..., N)$, $y_i^* = (y_{it_1}, ..., y_{it_r}, ..., y_{it_T})$, where each element of the vector of indices $(t_1, ..., t_T)$ is obtained by drawing with replacement from $(1, ..., T)$. Then, construct $Y^* = (y_1^*, ..., y_i^*, ...y_N^*)$. The same vector of indices is used to obtain X^*.

Cross-sectional and temporal resampling: The third scheme involves both cross-sectional and temporal resampling. Here, it first resamples from the cross-sectional dimension with probability $1/N$ (maintaining intact the temporal structure for each individual), and second, it resamples for each constructed cross-sectional unit across the time dimension with probability $1/T$.

3.2. Practical Implementation

For a given τ-quantile of interest, the bootstrapped panel data QR FE estimator is given by

$$(\hat{\eta}^*, \hat{\beta}^*) = \arg\min_{\eta,\beta} \sum_{i=1}^{N} \sum_{t=1}^{T} \rho_\tau(y_{it}^* - z_{it}^{*\prime}\eta - x_{it}^{*\prime}\beta),$$

where (y_{it}^*, x_{it}^*) is the pairwise bootstrap resampled data and z_{it}^* identifies the FE.

Implementation of the bootstrap in practice is simple. The steps for implementing the tests are as follows. Take B as a large integer. Given one of the three bootstraps discussed above, for each $b = 1, \ldots, B$:

(i) Obtain the resampled data $\{(y_{it}^b, z_{it}^b, x_{it}^b), i = 1, \ldots, n\ t = 1, \ldots, T\}$.
(ii) Estimate $\widehat{\beta}^b$.

The limiting distribution of $\hat{\beta}$ can be approximated by repeating these steps a large number of times, and the confidence intervals can be obtained by computing the quantiles of the implied empirical distribution using $\hat{\beta}^*$. More specifically, let $\hat{G}$ be the cumulative distribution function of a bootstrapped estimator. Let α be the level of the confidence interval. Then, one can compute the $1-\alpha$ percentile confidence interval for each element in the coefficient vector β by the $\alpha/2$ and $1-\alpha/2$ empirical percentiles of $\hat{G}$ as:

$$[\hat{\beta}^*_{\alpha/2}, \hat{\beta}^*_{1-\alpha/2}] = [\hat{G}^{-1}(\alpha/2), \hat{G}^{-1}(1-\alpha/2)].$$

4. Monte Carlo

In this section, we conduct a small Monte Carlo experiment to study and evaluate the finite sample properties of the proposed inference procedure. We evaluate the bootstrapping procedures for inference with the standard QR FE estimator. We use two different underlying models to generate the data: (i) random-effects (where the covariates are not related to the individual effects); and (ii) fixed-effects (where covariates are correlated with the individual effects) models. We compute empirical rejection rates of the confidence interval for the three different bootstrapping procedures described above.

4.1. Design

We follow the Monte Carlo framework of [1,2]. We evaluate the finite sample properties of the proposed inference procedure by computing the empirical rejection rates. We consider the following data-generating process:

$$y_{it} = \eta_i + x_{it}\beta + (\gamma_1 + \gamma_2 x_{it})u_{it} \quad i = 1, ..., N, \quad t = 1, ..., T. \tag{3}$$

Two simple versions of our basic model (3) are considered in the simulation experiments. In the first, the covariate x_{it} exerts a pure location shift effect, and we set $\gamma_1 = 1$ and $\gamma_2 = 0$. In the second, it has both location and scale shift effects, and we set $\gamma_1 = 1$ and $\gamma_2 = 0.1$. Note that in the location shift model, $\beta(\tau) = \beta$ for all τ, while in the second location and scale shift model, $\beta(\tau) = \beta + \gamma_2 F_u^{-1}(\tau)$, where F_u^{-1} is the inverse distribution function, or quantile, of u.

We consider different data-generating processes (DGPs). First, we consider three different distributions for η_i and u_{it}. In the first variant, both η_i and u_{it} are standard Gaussian; in the second, they are both $\chi^2_3 - 3$; in the third, they are both Cauchy.

The scalar covariate x is generated in two variants, with differences in each case for the location and location-scale shift models.

1. Random-effects model: $x_{it} = \theta_i + \epsilon_{it}$, where both θ_i and ϵ_{it} are standard Gaussian in the location shift model and χ^2_3 in the location-scale shift model[3]. This model has a covariate that is not correlated with the unobserved FE, and thus, the random-effects model is correct.
2. Fixed-effects model: $x_{it} = 0.3\eta_i + \epsilon_{it}$, where ϵ_{it} is standard Gaussian in the location shift model and χ^2_3 in the location-scale shift model. In this model, x is correlated with the unobserved FE.

We consider panel sizes of $N \in \{10, 25, 50\}$ and $T \in \{5, 10, 50\}$. Moreover, we consider bootstrap replications of size 500 and, for each experiment, compute all three types of bootstrap discussed in

[3] This is to avoid non-linearities in the linear quantile functions, which arise if x can take both positive and negative values.

Section 3: cross-sectional (CS), temporal (TE) and cross-sectional and temporal (CT). We set the number of Monte Carlo simulations to 500. The performance of the confidence interval bootstrap estimator is evaluated by computing the empirical rejection rate. Let $[\hat{\beta}^*_{\alpha/2}(\tau), \hat{\beta}^*_{1-\alpha/2}(\tau)]$ be the constructed $(1-\alpha)\times 100\%$ bootstrap confidence interval for $\beta(\tau)$ in a particular sample. Then, we compute the proportion of cases that $\beta(\tau) < \hat{\beta}^*_{\alpha/2}(\tau)$ or $\beta(\tau) > \hat{\beta}^*_{1-\alpha/2}(\tau)$ over the 500 Monte Carlo simulations, which is called the rejection rate. If the bootstrapping procedure correctly simulates the distribution of $\hat{\beta}(\tau)$, this proportion should be close to α. Finally, we consider $\alpha = 0.1$.

4.2. Results

Results for the case where the covariate x_{it} is not correlated with η are presented in Tables 1 and 2. The results for which the covariate x_{it} is correlated with η are presented in Tables 3 and 4.

Simulations results for the random-effects model (*i.e.*, x not correlated with η) appear in Tables 1 and 2 for the location and location-scale models, respectively. The best performance is observed for the CS bootstrapping procedure, where the confidence intervals coverage approaches 0.1 for sample sizes with $N = 25, 50$, although the coverage is too large for $N = 10$. TE produce slight under-coverage, that is confidence intervals that cover less than 0.1 for all cases of N and T. Note that CT has a particularly low coverage, close to 0.01, 1/10 of the target coverage. Normal, chi-squared and Cauchy DGPs show similar trends regarding the effect of both N and T, and the worst performance is for the Cauchy model. As suggested by an anonymous referee, the fact that TE and CT has a worse performance than CS could be due to the fact that conditional on the sample, the covariance terms present in the original variance are missing the TE or CT bootstrap, and thus, they do not mimic some features of the data.

The location model has better coverage properties than the location-scale one for $\tau = 0.25$ and $\tau = 0.75$, while they have similar properties for the median. The Cauchy model in which the density is very sparse at the corresponding first and third quartiles has a larger coverage than normal and chi-squared ones.

Note that the performance of the coverage for the estimated confidence intervals is not linear in T. That is, for a given N, increasing T does not monotonically increase or decrease coverage. In general, for a given N, a larger T results in a coverage closer to the nominal size of 0.1.

Simulations results for the fixed-effects model (*i.e.*, x correlated with η) appear in Tables 3 and 4 for the location and location-scale models, respectively. Similar results to those of the previous tables are observed. That is, the coverage is correct for the CS procedure, but requires $N = 25, 50$, while both the TE and CT produce are smaller than the nominal coverage. Note, however, that the location-scale in Table 4 shows a better performance than that of Table 2. This suggests that using information from the covariates helps in reducing the effect of the unobserved individual effects (*i.e.*, η).

Overall, the results confirm that, as expected, the empirical sizes approximate the nominal size as the sample size increases. In addition, the results show evidence that the CS procedure produces the closest coverage to the nominal size of 0.1, and this is achieved for large N (*i.e.*, ≥ 25) improving with larger T (in a non-linear relationship).

Table 1. Monte Carlo simulations, data-generating process (DGP): location model, random-effects model.

		Cross-Sectional			Temporal			Cross-Sectional and Temporal		
N	T	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$
					$\eta_i \sim N(0,1),\ u_{it} \sim N(0,1)$					
10	5	0.154	0.156	0.168	0.074	0.062	0.064	0.020	0.016	0.030
10	10	0.160	0.192	0.172	0.058	0.046	0.056	0.012	0.016	0.030
10	50	0.160	0.172	0.158	0.074	0.056	0.076	0.008	0.010	0.014
25	5	0.144	0.136	0.128	0.078	0.062	0.072	0.020	0.026	0.034
25	10	0.152	0.128	0.130	0.070	0.040	0.056	0.024	0.020	0.018
25	50	0.126	0.136	0.116	0.040	0.054	0.052	0.012	0.020	0.022
50	5	0.122	0.108	0.112	0.072	0.064	0.076	0.002	0.006	0.010
50	10	0.088	0.098	0.112	0.074	0.056	0.056	0.010	0.012	0.010
50	50	0.108	0.100	0.112	0.054	0.052	0.060	0.008	0.008	0.008
					$\eta_i \sim \chi^2_3,\ u_{it} \sim \chi^2_3$					
10	5	0.162	0.168	0.166	0.070	0.062	0.090	0.026	0.030	0.022
10	10	0.146	0.152	0.166	0.072	0.064	0.076	0.032	0.016	0.014
10	50	0.170	0.162	0.160	0.068	0.054	0.060	0.018	0.016	0.026
25	5	0.110	0.126	0.142	0.066	0.078	0.076	0.024	0.022	0.024
25	10	0.130	0.128	0.128	0.050	0.038	0.070	0.014	0.004	0.012
25	50	0.132	0.134	0.142	0.066	0.080	0.094	0.026	0.018	0.008
50	5	0.130	0.120	0.102	0.094	0.070	0.068	0.012	0.008	0.012
50	10	0.130	0.124	0.094	0.052	0.046	0.078	0.014	0.014	0.012
50	50	0.108	0.082	0.090	0.068	0.066	0.072	0.006	0.018	0.010
					$\eta_i \sim t_1,\ u_{it} \sim t_1$					
10	5	0.182	0.154	0.148	0.070	0.060	0.066	0.022	0.022	0.022
10	10	0.154	0.162	0.150	0.050	0.052	0.056	0.014	0.010	0.018
10	50	0.174	0.156	0.190	0.076	0.062	0.058	0.014	0.032	0.028
25	5	0.126	0.124	0.122	0.072	0.060	0.068	0.012	0.008	0.006
25	10	0.148	0.122	0.100	0.038	0.024	0.044	0.012	0.010	0.010
25	50	0.136	0.128	0.116	0.058	0.074	0.072	0.020	0.016	0.020
50	5	0.124	0.102	0.124	0.074	0.060	0.078	0.026	0.024	0.044
50	10	0.118	0.136	0.132	0.060	0.044	0.068	0.024	0.016	0.018
50	50	0.114	0.110	0.102	0.074	0.086	0.078	0.030	0.016	0.024

Notes: Rejection rates based on 500 replications. Monte Carlo experiments using 500 simulations. Theoretical size: 10%. See the text for details.

Table 2. Monte Carlo simulations, DGP: location-scale model, random-effects model.

		Cross-Sectional			Temporal			Cross-Sectional and Temporal		
N	T	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$
					$\eta_i \sim N(0,1),\ u_{it} \sim N(0,1)$					
10	5	0.190	0.178	0.180	0.130	0.088	0.076	0.016	0.016	0.032
10	10	0.158	0.174	0.158	0.090	0.074	0.088	0.020	0.030	0.042
10	50	0.150	0.136	0.154	0.096	0.074	0.072	0.030	0.018	0.018
25	5	0.148	0.132	0.140	0.122	0.080	0.130	0.054	0.028	0.042
25	10	0.122	0.116	0.134	0.094	0.058	0.088	0.030	0.012	0.022
25	50	0.118	0.114	0.122	0.096	0.082	0.100	0.016	0.028	0.022
50	5	0.138	0.150	0.152	0.136	0.064	0.154	0.048	0.020	0.066
50	10	0.120	0.128	0.092	0.098	0.052	0.076	0.024	0.016	0.034
50	50	0.102	0.128	0.122	0.082	0.076	0.090	0.026	0.016	0.034
					$\eta_i \sim \chi^2_3, u_{it} \sim \chi^2_3$					
10	5	0.178	0.190	0.186	0.108	0.088	0.124	0.040	0.028	0.046
10	10	0.136	0.152	0.152	0.060	0.054	0.098	0.014	0.012	0.018
10	50	0.196	0.154	0.160	0.076	0.060	0.082	0.036	0.044	0.034
25	5	0.146	0.138	0.148	0.078	0.090	0.148	0.022	0.018	0.068
25	10	0.118	0.136	0.128	0.080	0.062	0.090	0.020	0.016	0.038
25	50	0.138	0.130	0.140	0.094	0.064	0.086	0.024	0.018	0.024
50	5	0.136	0.122	0.152	0.096	0.100	0.240	0.034	0.026	0.100
50	10	0.106	0.110	0.126	0.094	0.080	0.100	0.018	0.012	0.050
50	50	0.128	0.086	0.120	0.082	0.064	0.080	0.020	0.016	0.016
					$\eta_i \sim t_1, u_{it} \sim t_1$					
10	5	0.210	0.178	0.196	0.126	0.064	0.084	0.044	0.022	0.040
10	10	0.210	0.166	0.174	0.092	0.072	0.112	0.044	0.020	0.046
10	50	0.172	0.160	0.166	0.074	0.076	0.082	0.020	0.020	0.018
25	5	0.158	0.118	0.144	0.100	0.064	0.132	0.044	0.020	0.056
25	10	0.128	0.124	0.150	0.088	0.046	0.096	0.050	0.026	0.044
25	50	0.126	0.130	0.148	0.072	0.076	0.068	0.016	0.020	0.030
50	5	0.172	0.146	0.128	0.148	0.040	0.136	0.048	0.010	0.056
50	10	0.116	0.112	0.130	0.142	0.060	0.114	0.020	0.002	0.022
50	50	0.154	0.102	0.148	0.086	0.084	0.102	0.016	0.018	0.024

Notes: Rejection rates based on 500 replications. Monte Carlo experiments using 500 simulations. Theoretical size: 10%. See the text for details.

Table 3. Monte Carlo simulations, DGP: location model, fixed-effects model.

		Cross-Sectional			Temporal			Cross-Sectional and Temporal		
N	T	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$
					$\eta_i \sim N(0,1),\ u_{it} \sim N(0,1)$					
10	5	0.174	0.156	0.148	0.082	0.060	0.068	0.034	0.026	0.030
10	10	0.152	0.172	0.152	0.058	0.032	0.038	0.022	0.012	0.022
10	50	0.140	0.148	0.188	0.076	0.072	0.082	0.024	0.018	0.012
25	5	0.110	0.108	0.122	0.080	0.066	0.082	0.016	0.010	0.016
25	10	0.136	0.136	0.138	0.060	0.048	0.076	0.014	0.014	0.024
25	50	0.116	0.126	0.112	0.060	0.070	0.072	0.016	0.010	0.006
50	5	0.126	0.112	0.108	0.094	0.052	0.072	0.026	0.022	0.020
50	10	0.106	0.114	0.112	0.056	0.038	0.040	0.010	0.020	0.020
50	50	0.136	0.132	0.102	0.090	0.084	0.072	0.020	0.020	0.010
					$\eta_i \sim \chi^2_3, u_{it} \sim \chi^2_3$					
10	5	0.188	0.172	0.158	0.088	0.072	0.070	0.008	0.010	0.012
10	10	0.182	0.160	0.160	0.064	0.068	0.076	0.016	0.022	0.026
10	50	0.156	0.160	0.128	0.074	0.072	0.078	0.014	0.016	0.016
25	5	0.124	0.122	0.112	0.098	0.066	0.062	0.032	0.032	0.030
25	10	0.156	0.136	0.126	0.056	0.044	0.066	0.022	0.008	0.012
25	50	0.140	0.140	0.124	0.058	0.072	0.062	0.016	0.012	0.016
50	5	0.100	0.108	0.146	0.078	0.058	0.072	0.016	0.014	0.016
50	10	0.096	0.110	0.110	0.044	0.026	0.050	0.012	0.012	0.012
50	50	0.122	0.110	0.088	0.064	0.064	0.084	0.018	0.020	0.016
					$\eta_i \sim t_1, u_{it} \sim t_1$					
10	5	0.176	0.166	0.154	0.090	0.078	0.096	0.028	0.018	0.016
10	10	0.130	0.144	0.176	0.056	0.052	0.056	0.014	0.006	0.014
10	50	0.190	0.166	0.146	0.072	0.060	0.062	0.028	0.014	0.018
25	5	0.128	0.090	0.110	0.072	0.042	0.070	0.016	0.024	0.030
25	10	0.120	0.118	0.108	0.052	0.062	0.062	0.016	0.012	0.010
25	50	0.136	0.146	0.130	0.080	0.084	0.064	0.020	0.020	0.018
50	5	0.130	0.134	0.110	0.072	0.058	0.068	0.024	0.022	0.028
50	10	0.132	0.120	0.104	0.052	0.044	0.056	0.020	0.012	0.010
50	50	0.120	0.118	0.122	0.060	0.052	0.070	0.012	0.014	0.020

Notes: Rejection rates based on 500 replications. Monte Carlo experiments using 500 simulations. Theoretical size: 10%. See the text for details.

Table 4. Monte Carlo simulations, DGP: location-scale model, fixed-effects model.

		Cross-Sectional			**Temporal**			**Cross-Sectional and Temporal**		
N	T	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$	$\tau = 0.25$	$\tau = 0.50$	$\tau = 0.75$
					$\eta_i \sim N(0,1),\ u_{it} \sim N(0,1)$					
10	5	0.166	0.150	0.184	0.102	0.074	0.122	0.044	0.036	0.032
10	10	0.206	0.170	0.164	0.100	0.066	0.096	0.020	0.012	0.038
10	50	0.172	0.150	0.188	0.080	0.080	0.104	0.030	0.018	0.016
25	5	0.130	0.116	0.144	0.142	0.074	0.128	0.030	0.012	0.038
25	10	0.126	0.118	0.126	0.094	0.052	0.086	0.038	0.020	0.044
25	50	0.152	0.118	0.134	0.078	0.070	0.068	0.010	0.010	0.010
50	5	0.130	0.150	0.132	0.162	0.076	0.146	0.046	0.022	0.048
50	10	0.132	0.112	0.134	0.106	0.076	0.140	0.030	0.010	0.018
50	50	0.112	0.110	0.122	0.078	0.062	0.076	0.020	0.014	0.026
					$\eta_i \sim \chi^2_3, u_{it} \sim \chi^2_3$					
10	5	0.184	0.142	0.172	0.084	0.084	0.130	0.028	0.040	0.072
10	10	0.158	0.162	0.186	0.072	0.054	0.098	0.014	0.020	0.050
10	50	0.188	0.156	0.180	0.086	0.086	0.096	0.026	0.032	0.032
25	5	0.158	0.138	0.146	0.112	0.078	0.190	0.024	0.032	0.072
25	10	0.130	0.150	0.112	0.086	0.052	0.100	0.020	0.014	0.040
25	50	0.110	0.104	0.106	0.070	0.076	0.072	0.034	0.016	0.012
50	5	0.136	0.130	0.166	0.102	0.084	0.254	0.032	0.028	0.086
50	10	0.120	0.132	0.136	0.078	0.072	0.150	0.016	0.008	0.036
50	50	0.122	0.122	0.116	0.066	0.086	0.094	0.020	0.014	0.014
					$\eta_i \sim t_1, u_{it} \sim t_1$					
10	5	0.200	0.164	0.170	0.108	0.048	0.130	0.034	0.014	0.024
10	10	0.150	0.138	0.184	0.098	0.058	0.106	0.034	0.018	0.038
10	50	0.158	0.148	0.138	0.104	0.086	0.106	0.026	0.022	0.024
25	5	0.140	0.140	0.146	0.146	0.082	0.136	0.050	0.020	0.050
25	10	0.132	0.136	0.142	0.110	0.038	0.102	0.036	0.016	0.026
25	50	0.100	0.106	0.142	0.104	0.066	0.092	0.034	0.008	0.036
50	5	0.156	0.104	0.146	0.192	0.068	0.178	0.072	0.014	0.064
50	10	0.118	0.116	0.120	0.138	0.052	0.124	0.044	0.012	0.036
50	50	0.122	0.148	0.140	0.098	0.076	0.108	0.036	0.020	0.008

Notes: Rejection rates based on 500 replications. Monte Carlo experiments using 500 simulations. Theoretical size: 10%. See the text for details.

5. Conclusions

The asymptotic variance-covariance matrix of the techniques developed in [1,2] for QR FE estimators cannot be easily obtained. They require estimation of the density of the innovations term. In order to overcome this difficulty, this paper explores bootstrap inference techniques for QR panel data. We propose to construct confidence intervals for the parameters of interest using percentile bootstrapping with a pairwise resampling technique. We evaluate the performance of the bootstrap confidence interval by means of Monte Carlo simulations. We assess the finite sample performance of the intervals by computing their empirical rejection rates. The results look promising and confirm that bootstrapping techniques can be used for inference in these models. In particular, we recommend using cross-section bootstrap with replacement from the cross-section dimension with probability $1/N$, maintaining intact the temporal structure for each individual i.

Acknowledgments

The authors would like to express their appreciation to the Editor, two anonymous referees, Anil Bera, Roger Koenker, Xuming He, Steve Portnoy and the participants in the seminars at the University of Illinois at Urbana-Champaign for helpful comments and discussions. All of the remaining errors are ours.

Author Contributions

The authors contributed equally to this work.

Conflicts of Interest

The authors declare no conflict of interest.

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